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APROACH TO NON-LINEAR FUZZY DIFFERENTIAL EQUATIONS USING WEEKLY COMPATIBLE SELF MAPPINGS IN NEUTROSOPHIC MERIC SPACES

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Abstract

In this article, demonstrated certain coincidence point and Common Fixed Point [CFP] findings under the rational type weakly compatible neutroshophic-contraction conditions [WNC-C] in complete neutroshophic metric spaces [CNM-spaces] utilizing the "triangular property of neutroshophic metric" with examples. Also provided neutroshophic non-linear fuzzy differential equations[NFDEs] as an application and demonstrated that the solution of the NFDEs has a unique CFP of the integral operators B, C and $\mathcal{g}$. This new path of weakly-compatible neutroshophic fuzzy-contraction [WNC-C] with the use of NFDEs in NM-space will be crucial. With different sorts of WNC-C for self-mappings and different forms of NFDEs in the context of NM-spaces, this approach may be expanded and developed in diverse directions.

1. Introduction

The Intuitionistic fuzzy metric space (IFM-space) set is introduced by Atanassov [1]. We followed L. Xia and Y.H. Tang [16] and [17] in establishing the FP theorem (also known as FPT) for IC-maps in FM-space. We then used this theorem in IFM-space and demonstrated FP in a similar manner. Next, neutrosophic metric spaces (NM-space), which address membership, non-membership, and naturalness functions, were presented by Kirişci and Simsek [7]. Some FPR in NM-space was demonstrated by Simsek and Kirişci [8] and Sowndrarajan et al. [10]. The neutrosophic soft linear spaces, often known as NSL-space, were defined by Bera and Mahapatra [2]. Neutropic soft normed linear spaces (NSNL-space), first presented by Mahapatra and Bera [3], have been around for a time. Neutropic norm, Cauchy sequence in NSNL-space, NSNL-space convexity, and metric in NSNL-space were all examined in [3]. Neutropic contractive mapping, or NC-mapping for short, was described by Kirisci et al. [6] and FP results in whole NM-space were demonstrated. Some of the following journals [4-5, 8-18] are very useful to following research work.

2. Preliminaries

Definition 2.1 [12]

An operation $*: [0,1]^2 \to [0,1]$ be a continuous t-norm, if

(i) $*$ be continuous

(ii) $*$ is commutative and associative

(iii) $1 * \varkappa_1 = \varkappa_1$ and $\varkappa_1 * \varkappa_2 \le \varkappa_3 * \varkappa_4$, whenever $\varkappa_1 \le \varkappa_3$ and $\varkappa_2 \le \varkappa_4, \forall \varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4 \in [0,1]$

An operation $\oplus : [0,1]^2 \to [0,1]$ is called a continuous t-conorm, if

(i) $\oplus$ be continuous

(ii) $\oplus$ is commutative and associative

(iii) $0 \oplus \varkappa_1 = \varkappa_1$ and $\varkappa_1 \oplus \varkappa_2 \le \varkappa_3 \oplus \varkappa_4$, whenever $\varkappa_1 \le \varkappa_3$ and $\varkappa_2 \le \varkappa_4$, $\forall \varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4 \in [0,1]$

The basic continuous t-conorms of maximum, product, and Lukasiewicz are defined by Schweizer and Sklar [30], respectively, as follows

(i) The minimum t-conorm is $\varkappa_1 * \varkappa_2 = \min\{\varkappa_1, \varkappa_2\}$

(ii) The product t-conorm is $\varkappa_1 * \varkappa_2 = \varkappa_1 \varkappa_2$

(iii) The Lukasiewicz t-norm is $\varkappa_1 \oplus \varkappa_2 = \max\{0, 1 - \varkappa_1, +\varkappa_2\}$

(iv) The minimum t-conorm is $\varkappa_1 \oplus \varkappa_2 = \max\{\varkappa_1, \varkappa_2\}$

(v) The product t-conorm is $\varkappa_1 \oplus \varkappa_2 = \varkappa_1, +\varkappa_2 - \varkappa_1 \varkappa_2$

(vi) The Lukasiewicz t-norm is $\varkappa_1 \oplus \varkappa_2 = \min\{\varkappa_1, +\varkappa_2, 1\}$

Definition 2.2 [15]

A 7-tuple $(\mathfrak{H}, \mathrm{M_r}, N_\mathrm{r}, R_\mathrm{r}, *, \oplus, \odot)$ be a NM-space if $\mathfrak{H}$ is an arbitrary set, $*$ be a continuous t-norm and $\oplus$ be continuous t-conorm and $(\mathrm{M_r}, N_\mathrm{r}, R_\mathrm{r})$ is neutrosophic set on $\mathfrak{H}^2 \times (0, \infty)$ satisfies:

($\mathfrak{R}$i) $\mathrm{M_r}(\varphi, \nu, t) \ge 0$

($\mathfrak{R}$ii) $\mathrm{M_r}(\varphi, \nu, t) = 1 \Leftrightarrow \varphi = \nu$

($\mathfrak{R}$iii) $\mathrm{M_r}(\varphi, \nu, t) = \mathrm{M_r}(\nu, \varphi, t)$

($\mathfrak{R}$iv) $\mathrm{M_r}(\varphi, \nu, t + s) \ge \mathrm{M_r}(\varphi, k, t) * \mathrm{M_r}(k, \nu, s)$

($\mathfrak{R}$v) $\mathrm{M_r}(\varphi, \nu, -): (0, \infty) \to [0,1]$ is continuous $\forall \varphi, \nu, k \in \mathfrak{H}$ and $t, s \in (0, \infty)$.

($\Re$vi) $\mathrm{N}_{\mathrm{r}}(\varphi, \nu, t) \leq 1$
($\Re\nu$ii) $\mathrm{N}_{\mathrm{r}}(\varphi, \nu, t) = 0 \Leftrightarrow \varphi = \nu$
($\Re\nu$iii) $\mathrm{N}_{\mathrm{r}}(\varphi, \nu, t) = \mathrm{N}_{\mathrm{r}}(\nu, \varphi, t)$
($\Re$ix) $\mathrm{N}_{\mathrm{r}}(\varphi, \nu, t+s) \leq N_{\mathrm{r}}(\varphi, k, t) \oplus \mathrm{N}_{\mathrm{r}}(k, \nu, s)$
($\Re$x) $\mathrm{N}_{\mathrm{r}}(\varphi, \nu, -): (0, \infty) \to [0,1]$ is continuous $\forall \varphi, \nu, k \in \mathfrak{H}$ and $t, s \in (0, \infty)$.
($\Re x$i) $\mathrm{R}_{\mathrm{r}}(\varphi, \nu, t) \leq 1$
($\Re$xii) $\mathrm{R}_{\mathrm{r}}(\varphi, \nu, t) = 0 \Leftrightarrow \varphi = \nu$
($\Re x$iii) $\mathrm{R}_{\mathrm{r}}(\varphi, \nu, t) = \mathrm{R}_{\mathrm{r}}(\nu, \varphi, t)$
($\Re x$iv) $\mathrm{R}_{\mathrm{r}}(\varphi, \nu, t+s) \leq \mathrm{R}_{\mathrm{r}}(\varphi, k, t) \odot \mathrm{R}_{\mathrm{r}}(k, \nu, s)$
($\Re x$v) $\mathrm{R}_{\mathrm{r}}(\varphi, \nu, -): (0, \infty) \to [0,1]$ is continuous $\forall \varphi, \nu, k \in \mathfrak{H}$ and $t, s \in (0, \infty)$. 1

Lemma 2.3 [13]

Let $\mathrm{N}_{\mathrm{r}}(\mathfrak{H}, x, *)$, $\mathrm{N}_{\mathrm{r}}(\mathfrak{H}, x, \oplus)$ and $\mathrm{N}_{\mathrm{r}}(\mathfrak{H}, x, \odot)$ is non-decreasing and non-increasing $\forall\ \varphi, \nu \in \mathfrak{H}$.

Definition 2.4 [14]

Let $(\mathfrak{H}, \mathrm{M}_{\mathrm{r}}, N_{\mathrm{r}}, R_{\mathrm{r}}, *, \oplus, \odot)$ be a neutrosophic space, $\varphi \in \mathfrak{H}$ and (φ_{j}) is a sequence in $\mathfrak{H}$. Then,

(i) (φ_{j}) converges to φ if $k \in (0,1)$and $t > 0. \exists\ \mathrm{j}_1 \in \aleph$,such that $\mathrm{M}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi, \mathrm{t}) > 1 - k, \forall\ j \geq \mathrm{j}_1$, $\mathrm{n}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi, \mathrm{t}) < k, \forall j \geq \mathrm{j}_1$, $\mathrm{R}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi, \mathrm{t}) < k, \forall j \geq \mathrm{j}_1$ We may write this $\lim_{j \to \infty} \varphi_{\mathrm{j}} = \varphi$ or $\varphi_{\mathrm{j}} \to \varphi$ as $\mathrm{j} \to \infty$.

(ii) (φ_{j}) is a Cauchy sequence[C-S] if $k \in (0,1)$ and $t > 0$. such that $\mathrm{j}_1 \in \aleph$,such that $\mathrm{M}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi_{\mathrm{i}}, \mathrm{t}) < 1 - k, N_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi_{\mathrm{i}}, \mathrm{t}) < k, \mathrm{R}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi_{\mathrm{i}}, \mathrm{t}) < k,, \forall j, i \geq \mathrm{j}_1$

(iii) $(\mathfrak{H}, \mathrm{M}_{\mathrm{r}}, N_{\mathrm{r}}, R_{\mathrm{r}}, *, \oplus, \odot)$ is complete if every C-S is convergent in $\mathfrak{H}$

(iv) N-C, if $\exists\ a \in (0, 1)$ and satisfying

$$\mathrm{M}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi_{\mathrm{j+1}}, \mathrm{t}) \geq a\left(\mathrm{M}_{\mathrm{r}}(\varphi_{\mathrm{j-1}}, \varphi_{\mathrm{j}}, \mathrm{t})\right) \text{ for } t > 0,\ j \geq 1$$
$$\mathrm{N}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi_{\mathrm{j+1}}, \mathrm{t}) \leq a\left(\mathrm{N}_{\mathrm{r}}(\varphi_{\mathrm{j-1}}, \varphi_{\mathrm{j}}, \mathrm{t})\right) \text{ for } t > 0,\ j \geq 1$$
$$\mathrm{R}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi_{\mathrm{j+1}}, \mathrm{t}) \leq a\left(R_{\mathrm{r}}(\varphi_{\mathrm{j-1}}, \varphi_{\mathrm{j}}, \mathrm{t})\right) \text{ for } t > 0,\ j \geq 1 \qquad 2$$

The set of natural numbers is consider this paper $\aleph$.

Lemma 2.5 [17]

Let $(\mathfrak{H}, \mathrm{M}_{\mathrm{r}}, N_{\mathrm{r}}, R_{\mathrm{r}}, *, \oplus, \odot)$ be a NM-space. A sequence φ_{j} in φ converges to $\varphi \in \mathfrak{H}$ if and only if $\mathrm{M}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi, \mathrm{t}) \to 1$, $\mathrm{N}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi, \mathrm{t}) \to 0$, and $\mathrm{R}_{\mathrm{r}}(\varphi_{\mathrm{j}}, \varphi, \mathrm{t}) \to 0$, as $j \to \infty$, for $t > 0$.

Definition 2.6

Let $(\mathfrak{H}, \mathrm{M}_{\mathrm{r}}, N_{\mathrm{r}}, R_{\mathrm{r}}, *, \oplus, \odot)$ be a NM-space. The NM-space $(\mathrm{M}_{\mathrm{r}}, N_{\mathrm{r}}, R_{\mathrm{r}})$ is triangular, if

$$\mathrm{M}_{\mathrm{r}}(\varphi, \nu, \mathrm{t}) \geq \mathrm{M}_{\mathrm{r}}(\varphi, k, \mathrm{t}) + \mathrm{M}_{\mathrm{r}}(k, \nu, \mathrm{t}) \forall \varphi, \nu, k \in \mathfrak{H}, t > 0.$$
$$\mathrm{N}_{\mathrm{r}}(\varphi, \nu, \mathrm{t}) \leq \mathrm{N}_{\mathrm{r}}(\varphi, k, \mathrm{t}) + \mathrm{N}_{\mathrm{r}}(k, \nu, \mathrm{t}) \forall \varphi, \nu, k \in \mathfrak{H}, t > 0.$$
$$\mathrm{R}_{\mathrm{r}}(\varphi, \nu, \mathrm{t}) \leq \mathrm{R}_{\mathrm{r}}(\varphi, k, \mathrm{t}) + \mathrm{R}_{\mathrm{r}}(k, \nu, \mathrm{t}) \forall \varphi, \nu, k \in \mathfrak{H}, t > 0. \qquad 3$$

Definition 2.7 [18]

Let $(\mathfrak{H}, \mathrm{M}_{\mathrm{r}}, N_{\mathrm{r}}, R_{\mathrm{r}}, *, \oplus, \odot)$ be a NM-space. A mapping $B: \mathfrak{H} \longrightarrow \mathfrak{H}$ is said to be a N-C if $\exists\ b \in (0,1)$ such that

$$\mathrm{M}_{\mathrm{r}}(\mathrm{B}_{\varphi}, \mathrm{B}_{\nu}, \mathrm{t}) \geq b(\mathrm{M}_{\mathrm{r}}(\varphi, \nu, \mathrm{t})) \forall\ \varphi, \nu \in \mathfrak{H}, t > 0.$$
$$\mathrm{N}_{\mathrm{r}}(\mathrm{B}_{\varphi}, \mathrm{B}_{\nu}, \mathrm{t}) \leq b(\mathrm{N}_{\mathrm{r}}(\varphi, \nu, \mathrm{t})) \forall\ \varphi, \nu \in \mathfrak{H}, t > 0.$$
$$R_{\mathrm{r}}(\mathrm{B}_{\varphi}, \mathrm{B}_{\nu}, \mathrm{t}) \leq b(R_{\mathrm{r}}(\varphi, \nu, \mathrm{t})) \forall\ \varphi, \nu \in \mathfrak{H}, t > 0. \qquad 4$$

Definition 2.8

Let $B$and g be two self-mappings on a nonempty set $\mathfrak{H}$ (i.e., $B, g: \mathfrak{H} \longrightarrow \mathfrak{H}$). If there exists $w \in \mathfrak{H}$ and $w = Bu = gu$ for some $u \in \mathfrak{H}$. Then, u is called a coincidence point of B and g, and w is called a point of coincidence of the mappings B *and* g. The mappings B *and* g are said to be weakly-compatible if they commute at their coincidence point, *i.e.*, $Bu = gu$ for some $u \in \mathfrak{H}$, then $Bgu = g\boldsymbol{u}$.

Proposition 2.9

Let B and g be weakly-compatible self-mappings on a nonempty set $\mathfrak{H}$. If B and g have a unique point of coincidence such that $w = Bu = gu$, then, u is known as the unique common NFP of B and ℓg.

3. Main Result

Throughout main results, we use the concept of a binary operation $*$ is a continuous product t-norm and $\oplus$ is a continuous product t-conorm which is defined as:

$$\varphi * \nu = \varphi \cdot v \text{ for all } \varphi, \nu \in [0, 1]$$
$$\varphi \oplus \nu = \varphi + v - \varphi \cdot v \text{ for all } \varphi, \nu \in [0, 1] \qquad 5$$

Theorem 3.1

Let a NM-space$(\mathrm{M}_{\mathrm{r}}, N_{\mathrm{r}}, R_{\mathrm{r}})$ is triangular in a CNM-space$(\mathfrak{H}, \mathrm{M}_{\mathrm{r}}, N_{\mathrm{r}}, R_{\mathrm{r}}, *, \oplus, \odot)$ and let$B, C, g: \mathfrak{H} \longrightarrow \mathfrak{H}$ be three self-mappings, satisfies, for all $\varphi, \nu \in \mathfrak{H}$,

$$\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\mathrm{B}_{\varphi},\mathrm{C}_{\mathrm{v}},\mathrm{t})}-1\right) \geq \left\{\begin{array}{c} e\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\mathcal{g}_{\varphi},\mathcal{g}_{\mathrm{v}},\mathrm{t})}-1\right)+f\left(\frac{1}{\frac{\mathrm{M}_{\mathrm{r}}(\mathcal{g}_{\mathrm{v}},B_{\varphi},2\mathrm{t}).\mathrm{M}_{\mathrm{r}}(\mathcal{g}_{\varphi},\mathrm{C}_{\mathrm{v}},2\mathrm{t})}{\mathrm{M}_{\mathrm{r}}(\mathcal{g}_{\varphi},\mathcal{g}_{\mathrm{v}},\mathrm{t})}}-1\right) \\ +g\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\mathcal{g}_{\varphi},B_{\varphi},\mathrm{t})}-1+\frac{1}{M_{\mathrm{r}}(\mathcal{g}_{\mathrm{v}},\mathrm{C}_{\mathrm{v}},\mathrm{t})}-1\right) \end{array}\right\},$$

$$\mathrm{N}_{\mathrm{r}}(\mathrm{B}_{\varphi},\mathrm{C}_{\mathrm{v}},\mathrm{t}) \leq \left\{\begin{array}{c} e\left(\mathrm{N}_{\mathrm{r}}(\mathcal{g}_{\varphi},\mathcal{g}_{\mathrm{v}},\mathrm{t})\right)+f\left(\frac{\mathrm{N}_{\mathrm{r}}(\mathcal{g}_{\mathrm{v}},B_{\varphi},2\mathrm{t}).\mathrm{N}_{\mathrm{r}}(\mathcal{g}_{\varphi},\mathrm{C}_{\mathrm{v}},2\mathrm{t})}{\mu_{\mathrm{r}}(\mathcal{g}_{\varphi},\mathcal{g}_{\mathrm{v}},\mathrm{t})}\right) \\ +g\left(N_{\mathrm{r}}(\mathcal{g}_{\varphi},B_{\varphi},\mathrm{t})+N_{\mathrm{r}}(\mathcal{g}_{\mathrm{v}},\mathrm{C}_{\mathrm{v}},\mathrm{t})\right) \end{array}\right\},$$

$$\mathrm{R}_{\mathrm{r}}(\mathrm{B}_{\varphi},\mathrm{C}_{\mathrm{v}},\mathrm{t}) \leq \left\{\begin{array}{c} e\left(\mathrm{R}_{\mathrm{r}}(\mathcal{g}_{\varphi},\mathcal{g}_{\mathrm{v}},\mathrm{t})\right)+f\left(\frac{\mathrm{R}_{\mathrm{r}}(\mathcal{g}_{\mathrm{v}},B_{\varphi},2\mathrm{t}).\mathrm{R}_{\mathrm{r}}(\mathcal{g}_{\varphi},\mathrm{C}_{\mathrm{v}},2\mathrm{t})}{\mu_{\mathrm{r}}(\mathcal{g}_{\varphi},\mathcal{g}_{\mathrm{v}},\mathrm{t})}\right) \\ +g\left(\mathrm{R}_{\mathrm{r}}(\mathcal{g}_{\varphi},B_{\varphi},\mathrm{t})+\mathrm{R}_{\mathrm{r}}(\mathcal{g}_{\mathrm{v}},\mathrm{C}_{\mathrm{v}},\mathrm{t})\right) \end{array}\right\}, \qquad 6$$

For $t > 0$ and $0 \leq e, f, g < 1$ with $(e + f + 2g) < 1$. If $B(\mathfrak{H}) \cup C(\mathfrak{H}) \subset \mathcal{g}(\mathfrak{H})$, where $\mathcal{g}(\mathfrak{H})$ is a complete subspace of $\mathfrak{H}$. Then B, C and $\mathcal{g}$ have a unique point of coincidence. Moreover, if the pairs $(B, \mathcal{g})$ and $(C, \mathcal{g})$ are weakly compatible. Then, B, C and $\mathcal{g}$ have a unique CFP in $\mathfrak{H}$.

Proof

Let φ_o be the arbitrary point of $\mathfrak{H}$. Using the condition $A(\mathfrak{H}) \cup B(\mathfrak{H}) \subset \mathcal{g}(\mathfrak{H})$ choose a sequence (φ_i) in $\mathfrak{H}$ such that

$\mathcal{g}\varphi_{2i+1} = \mathrm{B}\varphi_{2i}$ and $\mathcal{g}\varphi_{2i+2} = C\varphi_{2i+1}$, for all $i \geq 0$. 7

Now, by (6), for $t > 0$,

$$\frac{1}{\mathrm{M}_{\mathrm{r}}(\mathcal{g}\varphi_{2i+1},\mathcal{g}\varphi_{2i+2},\mathrm{t})}-1=\frac{1}{\mathrm{M}_{\mathrm{r}}(A\varphi_{2i},\mathrm{B}\varphi_{2i+1},\mathrm{t})}-1$$

$$\geq \left\{\begin{array}{c} a\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+1},\mathrm{t})}-1\right) \\ +b\left(\frac{1}{\frac{\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i+1},Bw_{2i},2\mathrm{t}).\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i},Cw_{2i+1},2\mathrm{t})}{\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+1},\mathrm{t})}}-1\right) \\ +c(\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i},Bw_{2i},\mathrm{t})+\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i+1},Cw_{2i+1},\mathrm{t})) \end{array}\right\}$$

$$= \left\{\begin{array}{c} a\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+1},\mathrm{t})}-1\right) \\ +b\left(\frac{1}{\frac{\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i+1},\mathcal{g}w_{2i+1},2\mathrm{t}).\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+2},2\mathrm{t})}{\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+1},\mathrm{t})}}-1\right) \\ +c\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+1},\mathrm{t})}-1+\frac{1}{\mathrm{M}_{\mathrm{r}}(\mathcal{g}w_{2i+1},\mathcal{g}w_{2i+2},\mathrm{t})}-1\right) \end{array}\right\}$$

$$= \left\{\begin{array}{c} a\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i},\ell w_{2i+1},\mathrm{t})}-1\right) \\ +b\left(\frac{1}{\frac{\mathrm{M}_{\mathrm{r}}(\ell w_{2i},\ell w_{2i+2},2\mathrm{t})}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i},\ell w_{2i+1},\mathrm{t})M_{\mathrm{r}}(\ell w_{2i},\ell w_{2i+2},2\mathrm{t})}}-1\right) \\ +c\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i},lw_{2i+1},\mathrm{t})}-1+\frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i+1},lw_{2i+2},\mathrm{t})}-1\right) \end{array}\right\}$$

$$N_{\mathrm{r}}(\mathcal{g}\varphi_{2i+1},\mathcal{g}\varphi_{2i+2},\mathrm{t}) = \mathrm{N}_{\mathrm{r}}(A\varphi_{2i},\mathrm{B}\varphi_{2i+1},\mathrm{t})$$

$$\geq \left\{\begin{array}{c} a\left(N_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+1},\mathrm{t})\right) \\ +b\left(\frac{\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i+1},Bw_{2i},2\mathrm{t}).\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i},Cw_{2i+1},2\mathrm{t})}{\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+1},\mathrm{t})}\right) \\ +c(\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i},Bw_{2i},\mathrm{t})+\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i+1},Cw_{2i+1},\mathrm{t})) \end{array}\right\}$$

$$= \left\{\begin{array}{c} a\left(\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+1},\mathrm{t})\right) \\ +b\left(\frac{\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i+1},\mathcal{g}w_{2i+1},2\mathrm{t}).\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+2},2\mathrm{t})}{\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+1},\mathrm{t})}\right) \\ +c(\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i},\mathcal{g}w_{2i+1},\mathrm{t})+\mathrm{N}_{\mathrm{r}}(\mathcal{g}w_{2i+1},\mathcal{g}w_{2i+2},\mathrm{t})) \end{array}\right\}$$

$$= \begin{Bmatrix} a\left(\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, \ell w_{2i+1}, \mathrm{t})\right) \\ +b\left(\frac{\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, \ell w_{2i+2}, 2\mathrm{t})}{\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, \ell w_{2i+1}, \mathrm{t})\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, \ell w_{2i+2}, 2\mathrm{t})}\right) \\ +c(N_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}) + N_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})) \end{Bmatrix}$$

$$\mathrm{R}_{\mathrm{r}}(g\varphi_{2i+1}, g\varphi_{2i+2}, \mathrm{t}) = \mathrm{R}_{\mathrm{r}}(A\varphi_{2i}, \mathrm{B}\varphi_{2i+1}, \mathrm{t})$$

$$\geq \begin{Bmatrix} a\left(R_{\mathrm{r}}(gw_{2i}, gw_{2i+1}, \mathrm{t})\right) \\ +b\left(\frac{R_{\mathrm{r}}(gw_{2i+1}, Bw_{2i}, 2\mathrm{t}).R_{\mathrm{r}}(gw_{2i}, Cw_{2i+1}, 2\mathrm{t})}{\mathrm{R}_{\mathrm{r}}(gw_{2i}, gw_{2i+1}, \mathrm{t})}\right) \\ +c(\mathrm{R}_{\mathrm{r}}(gw_{2i}, Bw_{2i}, \mathrm{t}) + R_{\mathrm{r}}(gw_{2i+1}, Cw_{2i+1}, \mathrm{t})) \end{Bmatrix}$$

$$= \begin{Bmatrix} a\left(\mathrm{R}_{\mathrm{r}}(gw_{2i}, gw_{2i+1}, \mathrm{t})\right) \\ +b\left(\frac{\mathrm{R}_{\mathrm{r}}(gw_{2i+1}, gw_{2i+1}, 2\mathrm{t}).\mathrm{R}_{\mathrm{r}}(gw_{2i}, gw_{2i+2}, 2\mathrm{t})}{\mathrm{R}_{\mathrm{r}}(gw_{2i}, gw_{2i+1}, \mathrm{t})}\right) \\ +c(\mathrm{R}_{\mathrm{r}}(gw_{2i}, gw_{2i+1}, \mathrm{t}) + \mathrm{R}_{\mathrm{r}}(gw_{2i+1}, gw_{2i+2}, \mathrm{t})) \end{Bmatrix}$$

$$= \begin{Bmatrix} a\left(\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, \ell w_{2i+1}, \mathrm{t})\right) \\ +b\left(\frac{\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, \ell w_{2i+2}, 2\mathrm{t})}{\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, \ell w_{2i+1}, \mathrm{t})\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, \ell w_{2i+2}, 2\mathrm{t})}\right) \\ +c(\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}) + \mathrm{R}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})) \end{Bmatrix} \quad 8$$

From, Definition 2.2 (iv, ix, xiv),

$\mathrm{M}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+2}, 2\mathrm{t}) \geq M_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}) * M_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})$ for $\mathrm{t} > 0$

$$\frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})} - 1 \geq \begin{Bmatrix} a\left(\frac{1}{M_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})} - 1\right) \\ +b\left(\frac{1}{\frac{\mathrm{M}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}).\mathrm{M}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})}{M_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})}} - 1\right) \\ +c\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})} - 1 + \frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})} - 1\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} a\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})} - 1\right) + b\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})} - 1\right) \\ +c\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})} - 1 + \frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})} - 1\right) \end{Bmatrix}$$

$$= (a + c)\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})} - 1\right) + (b + c)\left(\frac{1}{\mathrm{M}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})} - 1\right)$$

$\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+2}, 2\mathrm{t}) \leq \mathrm{N}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}) \oplus \mathrm{N}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})$ for $\mathrm{t} > 0$

$$N_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t}) \leq \begin{Bmatrix} a\left(\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\right) \\ +b\left(\frac{\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}).N_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})}{N_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})}\right) \\ +c\left(\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}) + N_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} a\left(\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\right) + b\left(\mathrm{N}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\right) \\ +c\left(\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}) + \mathrm{N}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\right) \end{Bmatrix}$$

$$= (a + c)\left(N_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\right) + (b + c)\left(\mathrm{N}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\right)$$

$\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+2}, 2\mathrm{t}) \leq \mathrm{R}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}) \odot R_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})$ for $\mathrm{t} > 0$

$$\mathrm{R}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t}) \leq \begin{Bmatrix} a\left(\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\right) \\ +b\left(\frac{\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}).\mathrm{R}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})}{\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})}\right) \\ +c\left(\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}) + \mathrm{R}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} a\left(\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\right) + b\left(\mathrm{R}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\right) \\ +c\left(\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t}) + \mathrm{R}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\right) \end{Bmatrix}$$

$$= (a + c)\left(R_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\right) + (b + c)\left(\mathrm{R}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\right) \quad 3.9$$

After simplification, we obtain

$\mathrm{M}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t}) \geq Y\left(\mathrm{M}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\right), for\ t > 0,$

$\mathrm{N}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t}) \leq Y\left(\mathrm{N}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\right), for\ t > 0,$

$\mathrm{R}_{\mathrm{r}}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t}) \leq Y\left(\mathrm{R}_{\mathrm{r}}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\right), for\ t > 0,$ 10

where $Y = \frac{a + c}{(1 - b - c)} < 1$.

Similarly, from (6), for $t > 0$,

$\mathrm{M}_{\mathrm{r}}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t}) = \mathrm{M}_{\mathrm{r}}(Aw_{2i+2}\mathrm{B}lw_{2i+1}, \mathrm{t})$

$$\leq \left\{ \begin{array}{c} a\left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+1}, t)} - 1\right) \\ +b\left(\frac{M_r(\ell w_{2i+1}, Aw_{2i+2}2, t).M_r(\ell w_{2i+2}, Bw_{2i+1}, t)}{M_r(\ell w_{2i+2}, lw_{2i+1}, t)} - 1\right) \\ +c\left(\frac{1}{M_r(\ell w_{2i+2}, Aw_{2i+2}, t)} - 1 + \frac{1}{M_r(\ell w_{2i+1}, Bw_{2i+1}, t)} - 1\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} a\left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+1}, t)} - 1\right) + b\frac{M_r(\ell w_{2i+2}, lw_{2i+1}, t)}{M_r(\ell w_{2i+2}, lw_{2i+3}, t)} - 1 + \\ c\left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+3}, t)} - 1 + \frac{1}{M_r(\ell w_{2i+1}, lw_{2i+2}, t)} - 1\right) \end{array} \right\}$$

$N_r(\ell w_{2i+2}, lw_{2i+3}, t) = N_r(Aw_{2i+2}Blw_{2i+1}, t)$

$$\leq \left\{ \begin{array}{c} a\left(N_r(\ell w_{2i+2}, lw_{2i+1}, t)\right) \\ +b\left(\frac{N_r(\ell w_{2i+1}, Aw_{2i+2}2, t).N_r(\ell w_{2i+2}, Bw_{2i+1}, t)}{N_r(\ell w_{2i+2}, lw_{2i+1}, t)}\right) \\ +c\left(N_r(\ell w_{2i+2}, Aw_{2i+2}, t) + N_r(\ell w_{2i+1}, Bw_{2i+1}, t)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} a\left(N_r(\ell w_{2i+2}, lw_{2i+1}, t)\right) \\ +b\left(\frac{N_r(\ell w_{2i+1}, lw_{2i+3}2, t).N_r(\ell w_{2i+2}, lw_{2i+2}, 2t)}{N_r(\ell w_{2i+2}, lw_{2i+1}, t)}\right) \\ +c\left(N_r(\ell w_{2i+2}, lw_{2i+3}, t) + N_r(\ell w_{2i+1}, lw_{2i+2}, t)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} a\left(N_r(\ell w_{2i+2}, lw_{2i+1}, t)\right) + b\left(\frac{N_r(\ell w_{2i+2}, lw_{2i+1}, t)}{N_r(\ell w_{2i+2}, lw_{2i+3}, t)}\right) \\ c\left(N_r(\ell w_{2i+2}, lw_{2i+3}, t) + N_r(\ell w_{2i+1}, lw_{2i+2}, t)\right) \end{array} \right\}$$

$R_r(\ell w_{2i+2}, lw_{2i+3}, t) = R_r(Aw_{2i+2}Blw_{2i+1}, t)$

$$\leq \left\{ \begin{array}{c} a\left(R_r(\ell w_{2i+2}, lw_{2i+1}, t)\right) \\ +b\left(\frac{R_r(\ell w_{2i+1}, Aw_{2i+2}2, t).R_r(\ell w_{2i+2}, Bw_{2i+1}, t)}{R_r(\ell w_{2i+2}, lw_{2i+1}, t)}\right) \\ +c\left(R_r(\ell w_{2i+2}, Aw_{2i+2}, t) + R_r(\ell w_{2i+1}, Bw_{2i+1}, t)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} a\left(R_r(\ell w_{2i+2}, lw_{2i+1}, t)\right) \\ +b\left(\frac{R_r(\ell w_{2i+1}, lw_{2i+3}2, t).R_r(\ell w_{2i+2}, lw_{2i+2}, 2t)}{R_r(\ell w_{2i+2}, lw_{2i+1}, t)}\right) \\ +c\left(R_r(\ell w_{2i+2}, lw_{2i+3}, t) + R_r(\ell w_{2i+1}, lw_{2i+2}, t)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} a\left(R_r(\ell w_{2i+2}, lw_{2i+1}, t)\right) + b\left(\frac{M_r(\ell w_{2i+2}, lw_{2i+1}, t)}{M_r(\ell w_{2i+2}, lw_{2i+3}, t)}\right) \\ +c\left(R_r(\ell w_{2i+2}, lw_{2i+3}, t) + R_r(\ell w_{2i+1}, lw_{2i+2}, t)\right) \end{array} \right\} \quad 11$$

From, Definition 2.2 (iv, ix, xiv),

$$\frac{1}{M_r(\ell w_{2i+1}, lw_{2i+3}, 2t)} - 1 \geq \left(\frac{1}{M_r(\ell w_{2i+1}, lw_{2i+2}, t)} - 1\right) * \left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+3}, t)} - 1\right)$$

$$\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+3}, t)} - 1 \quad \leq \left\{ \begin{array}{c} a\left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+1}, t)} - 1\right) \\ +b\left(\frac{\left(\frac{1}{M_r(\ell w_{2i+1}, lw_{2i+2}, t)} - 1\right).\left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+3}, t)} - 1\right)}{\left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+1}, t)} - 1\right)}\right) + \\ c\left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+3}, t)} - 1 + \frac{1}{M_r(\ell w_{2i+1}, lw_{2i+2}, t)} - 1\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} a\left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+1}, t)} - 1\right) + b\left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+3}, t)} - 1\right) \\ +c\left(M_r(\ell w_{2i+2}, lw_{2i+3}, t) + M_r(\ell w_{2i+1}, lw_{2i+2}, t)\right) \end{array} \right\}$$

$$= (a+c)\left(\frac{1}{M_r(\ell w_{2i+1}, lw_{2i+2}, t)} - 1\right) + (b+c)\left(\frac{1}{M_r(\ell w_{2i+2}, lw_{2i+3}, t)} - 1\right)$$

$N_r(\ell w_{2i+1}, lw_{2i+3}, 2t) \leq N_r(\ell w_{2i+1}, lw_{2i+2}, t) \oplus N_r(\ell w_{2i+2}, lw_{2i+3}, t)$ for $t > 0$

$$N_r(\ell w_{2i+2}, lw_{2i+3}, t) \leq \left\{ \begin{array}{c} a\left(N_r(\ell w_{2i+2}, lw_{2i+1}, t)\right) \\ +b\left(\frac{N_r(\ell w_{2i+1}, lw_{2i+2}, t).N_r(\ell w_{2i+2}, lw_{2i+3}, t)}{N_r(\ell w_{2i+2}, lw_{2i+1}, t)}\right) + \\ c\left(N_r(\ell w_{2i+2}, lw_{2i+3}, t) + N_r(\ell w_{2i+1}, lw_{2i+2}, t)\right) \end{array} \right\}$$

$$= \begin{Bmatrix} a\big(\mathrm{N_r}(\ell w_{2i+2}, lw_{2i+1}, \mathrm{t})\big) + b\big(\mathrm{N_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t})\big) \\ +c\big(\mathrm{N_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t}) + \mathrm{N_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\big) \end{Bmatrix}$$

$$= (a+c)\big(\mathrm{N_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\big) + (b+c)\big(\mathrm{N_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t})\big)$$

$\mathrm{R_r}(\ell w_{2i+1}, lw_{2i+3}, 2\mathrm{t}) \leq \mathrm{R_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t}) \odot \mathrm{R_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t})$ for $\mathrm{t} > 0$

$$\mathrm{R_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t}) \leq \begin{Bmatrix} a\big(R_\mathrm{r}(\ell w_{2i+2}, lw_{2i+1}, \mathrm{t})\big) \\ +b\left(\dfrac{\mathrm{R_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t}).R_\mathrm{r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t})}{\mathrm{R_r}(\ell w_{2i+2}, lw_{2i+1}, \mathrm{t})}\right) + \\ c\big(\mathrm{R_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t}) + \mathrm{R_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\big) \end{Bmatrix}$$

$$= \begin{Bmatrix} a\big(\mathrm{R_r}(\ell w_{2i+2}, lw_{2i+1}, \mathrm{t})\big) + b\big(\mathrm{R_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t})\big) \\ +c\big(\mathrm{R_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t}) + \mathrm{R_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\big) \end{Bmatrix}$$

$$= (a+c)\big(\mathrm{R_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\big) + (b+c)\big(\mathrm{R_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t})\big) \quad 12$$

After simplification, we obtain,

$$\frac{1}{\mathrm{M_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t})} - 1 \leq Y\left(\frac{1}{\mathrm{M_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})} - 1\right) for\ t > 0,$$

$$\mathrm{N_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t}) \leq Y\big(\mathrm{N_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\big)\ for\ t > 0,$$

$$\mathrm{R_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t}) \leq Y\big(\mathrm{R_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\big)\ for\ t > 0, \quad 13$$

Where Y value is same as in (10). From (10), (13), and by induction.

$$\frac{1}{\mathrm{M_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t})} - 1 \leq Y\left(\frac{1}{\mathrm{M_r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})} - 1\right) \leq Y^2\left(\frac{1}{\mathrm{M_r}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})} - 1\right)$$
$$\leq \cdots \leq Y^{2i+2}\big(\mathrm{M_r}(\ell w_0, lw_1, \mathrm{t})\big) \to 1\ as\ i \to \infty.$$

$$\mathrm{N_r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t}) \leq Y\big(N_\mathrm{r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\big) \leq Y^2\big(\mathrm{N_r}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\big)$$
$$\leq \cdots \leq Y^{2i+2}\big(\mathrm{N_r}(\ell w_0, lw_1, \mathrm{t})\big) \to 0\ as\ i \to \infty.$$

$$R_\mathrm{r}(\ell w_{2i+2}, lw_{2i+3}, \mathrm{t}) \leq Y\big(R_\mathrm{r}(\ell w_{2i+1}, lw_{2i+2}, \mathrm{t})\big) \leq Y^2\big(R_\mathrm{r}(\ell w_{2i}, lw_{2i+1}, \mathrm{t})\big)$$
$$\leq \cdots \leq Y^{2i+2}\big(R_\mathrm{r}(\ell w_0, lw_1, \mathrm{t})\big) \to 0\ as\ i \to \infty. \quad 14$$

Hence, $(lw_i)_{i\geq 0}$ is a NC-sequence in$(\mathfrak{H}, \mathrm{M_r}, N_\mathrm{r}, R_\mathrm{r}, *, \oplus, \odot)$, therefore,

$$\lim_{i\to\infty} M_r\big(lw_i, (lw_{i-1}, t)\big) = 1\ for\ t > 0,$$

$$\lim_{i\to\infty} N_r\big(lw_i, (lw_{i-1}, t)\big) = 0\ for\ t > 0.$$

$$\lim_{i\to\infty} R_r\big(lw_i, (lw_{i-1}, t)\big) = 0\ for\ t > 0 \quad 15$$

Since (M_r, N_r, R_r) is triangular, $j > i > i_0$,

$$M_r\left(lw_i, \left(lw_j, t\right)\right) \leq \begin{Bmatrix} \big(M_r\big(lw_i, (lw_{i+1}, t)\big)\big) + \big(M_r\big(lw_{i+1}, (lw_{i+2}, t)\big)\big) \\ + \cdots + \left(M_r\left(lw_{i-1}, \left(lw_j, t\right)\right)\right) \end{Bmatrix}$$
$$\leq (Y^i + Y^{i+1} + \cdots + Y^{j-1}).(M_r(lw_o, lw_1, t))$$
$$\leq \left(\frac{Y^i}{1-Y}\right)(M_r(lw_o, lw_1, t)) \to 0, as\ i \to \infty.$$

$$N_r\left(lw_i, \left(lw_j, t\right)\right) \leq \begin{Bmatrix} \big(N_r\big(lw_i, (lw_{i+1}, t)\big)\big) + \big(N_r\big(lw_{i+1}, (lw_{i+2}, t)\big)\big) \\ + \cdots + \left(N_r\left(lw_{i-1}, \left(lw_j, t\right)\right)\right) \end{Bmatrix}$$
$$\leq \begin{Bmatrix} Y^i\big(M_r\big(lw_i, (lw_{i+1}, t)\big)\big) + Y^{i+1}\big(N_r\big(lw_{i+1}, (lw_{i+2}, t)\big)\big) \\ + \cdots + Y^{j-1}\left(N_r\left(lw_{i-1}, \left(lw_j, t\right)\right)\right) \end{Bmatrix}$$
$$\leq (Y^i + Y^{i+1} + \cdots + Y^{j-1}).(N_r(lw_o, lw_1, t))$$
$$\leq \left(\frac{Y^i}{1-Y}\right)(N_r(lw_o, lw_1, t)) \to 0, as\ i \to \infty.$$

$$R_r\left(lw_i, \left(lw_j, t\right)\right) \leq \begin{Bmatrix} \big(R_r\big(lw_i, (lw_{i+1}, t)\big)\big) + \big(R_r\big(lw_{i+1}, (lw_{i+2}, t)\big)\big) \\ + \cdots + \left(R_r\left(lw_{i-1}, \left(lw_j, t\right)\right)\right) \end{Bmatrix}$$
$$\leq \begin{Bmatrix} Y^i\big(R_r\big(lw_i, (lw_{i+1}, t)\big)\big) + Y^{i+1}\big(R_r\big(lw_{i+1}, (lw_{i+2}, t)\big)\big) \\ + \cdots + Y^{j-1}\left(R_r\left(lw_{i-1}, \left(lw_j, t\right)\right)\right) \end{Bmatrix}$$
$$\leq (Y^i + Y^{i+1} + \cdots + Y^{j-1}).(R_r(lw_o, lw_1, t))$$
$$\leq \left(\frac{Y^i}{1-Y}\right)(R_r(lw_o, lw_1, t)) \to 0, as\ i \to \infty. \quad 16$$

This shows that (lw_i) is a C-S, and $l(w)$ is a complete subspace of W. Hence, $\exists\, u, v \in W$ such that $\ell wi \longrightarrow u = \ell v\ as\ i \longrightarrow \infty, i.e.,$

$$\lim_{i\to\infty} M_r\,(u, lw_i, t) = M_r(u, lv, t) = 1\ for\ t > 0,$$

$\lim_{i\to\infty} N_r(u, lw_i, t) = N_r(u, lv, t) = 0\ for\ t > 0,$
$\lim_{i\to\infty} R_r(u, lw_i, t) = R_r(u, lv, t) = 0\ for\ t > 0.$ 17

Since (M_r, N_r, R_r) is triangular,
$M_r(lv, Av, t) \leq (M_r(lv, lw_{2i+2}, t)) + (M_r(lw_{2i+2}, Av, t))\ for\ t > 0,$
$N_r(lv, Av, t) \leq (N_r(lv, lw_{2i+2}, t)) + (N_r(lw_{2i+2}, Av, t))\ for\ t > 0,$
$R_r(lv, Av, t) \leq (R_r(lv, lw_{2i+2}, t)) + (R_r(lw_{2i+2}, Av, t))\ for\ t > 0.$ 18

Now get (6), (15), (17), and from, Definition 2.2 (iv, ix, xiv), for $t > 0$,

$$\frac{1}{M_r(lw_{2i+2}, Av, t)} - 1 = \left(\frac{1}{M_r(Av, Bw_{2i+1}, t)} - 1\right)$$

$$\leq \left\{\begin{matrix} a\left(\frac{1}{M_r(\ell v, lw_{2i+1}, t)} - 1\right) \\ +b\left(\frac{M_r(\ell w_{2i+1}, lv, t).M_r(\ell v, Av, t).M_r(\ell v, lw_{2i+2}, 2t)}{M_r(\ell v, lw_{2i+1}, t)} - 1\right) \\ +c\left(\frac{1}{M_r(\ell v, Av, t)} - 1 + \frac{1}{M_r(\ell w_{2i+1}, lw_{2i+2}, t)} - 1\right) \end{matrix}\right\}$$
$$\to (b+c)\left(\frac{1}{M_r(\ell v, Av, t)} - 1\right)\ as\ i \to \infty$$

$$N_r(lw_{2i+2}, Av, t) = (N_r(Av, Bw_{2i+1}, t))$$
$$\leq \left\{\begin{matrix} a\left(N_r(\ell v, lw_{2i+1}, t)\right) + b\left(\frac{N_r(\ell w_{2i+1}, Av, 2t).N_r(\ell v, Bw_{2i+1}, 2t)}{N_r(\ell v, lw_{2i+1}, t)}\right) \\ +c\left(N_r(\ell v, Av, t) + N_r(\ell w_{2i+1}, Bw_{2i+1}, t)\right) \end{matrix}\right\}$$
$$\leq \left\{\begin{matrix} a\left(N_r(\ell v, lw_{2i+1}, t)\right) \\ +b\left(\frac{N_r(\ell w_{2i+1}, lv, t).N_r(\ell v, Av, t).N_r(\ell v, lw_{2i+2}, 2t)}{N_r(\ell v, lw_{2i+1}, t)}\right) \\ +c\left(N_r(\ell v, Av, t) + N_r(\ell w_{2i+1}, lw_{2i+2}, t)\right) \end{matrix}\right\}$$
$$\to (b+c)\left(N_r(\ell v, Av, t)\right)\ as\ i \to \infty$$

$$R_r(lw_{2i+2}, Av, t) = (R_r(Av, Bw_{2i+1}, t))$$

$$\leq \left\{\begin{matrix} a\left(R_r(\ell v, lw_{2i+1}, t)\right) \\ +b\left(\frac{R_r(\ell w_{2i+1}, lv, t).R_r(\ell v, Av, t).R_r(\ell v, lw_{2i+2}, 2t)}{R_r(\ell v, lw_{2i+1}, t)}\right) \\ +c\left(R_r(\ell v, Av, t) + R_r(\ell w_{2i+1}, lw_{2i+2}, t)\right) \end{matrix}\right\}$$
$$\to (b+c)\left(R_r(\ell v, Av, t)\right)\ as\ i \to \infty.$$ 19

Then,
$\lim_{i\to\infty} \sup\left(M_r(\ell w_{2i+2}, Av, t)\right) \geq (b+c)\left(M_r(\ell v, Av, t)\right)\ for\ t > 0,$
$\lim_{i\to\infty} \inf\left(M_r(\ell w_{2i+2}, Av, t)\right) \leq (b+c)\left(M_r(\ell v, Av, t)\right)\ for\ t > 0\ ,$
$\lim_{i\to\infty} \inf\left(R_r(\ell w_{2i+2}, Av, t)\right) \leq (b+c)\left(R_r(\ell v, Av, t)\right)\ for\ t > 0,$ 20

From (17), (18), and (20), we get
$\frac{1}{M_r(\ell v, Av, t)} - 1 \geq (b+c)\left(\frac{1}{M_r(\ell v, Av, t)} - 1\right)\ for\ t > 0,$
$N_r(\ell v, Av, t) \leq (b+c)\left(N_r(\ell v, Av, t)\right)\ for\ t > 0,$
$R_r(\ell v, Av, t) \leq (b+c)\left(R_r(\ell v, Av, t)\right)\ for\ t > 0.$ 21

Note that $(b+c) < 1$, where $(a+b+2c) < 1$, therefore,
$\frac{1}{M_r(\ell v, Av, t)} - 1 = \frac{1}{M_r(u, Av, t)} - 1 = 1 \Rightarrow u$ = lv = Av, for $t > 0$.
$N_r(\ell v, Av, t) = N_r(u, Av, t) = 0 \Rightarrow u$ = lv = Av, for $t > 0$.
$R_r(\ell v, Av, t) = R_r(u, Av, t) = 0 \Rightarrow u$ = lv = Av, for $t > 0$.

Next, prove that $u = \ell v = Bv$. Since, (M_r, N_r, R_r) is triangular.
$\frac{1}{M_r(lv, Bv, t)} - 1 = \frac{1}{M_r(lv, lw_{2i+1}, t)} - 1 + \frac{1}{M_r(lw_{2i+1}, Bv, t)} - 1 \quad for\ t > 0,$
$N_r(lv, Bv, t) = N_r(lv, lw_{2i+1}, t) + N_r(lw_{2i+1}, Bv, t) \quad for\ t > 0,$
$R_r(lv, Bv, t) = R_r(lv, lw_{2i+1}, t) + R_r(lw_{2i+1}, Bv, t) \quad for\ t > 0.$ 22

Now, again from (6), (15), (17), and from, Definition 2.2 (iv, ix, xiv), for $t > 0$,

$$\left(M_r(lw_{2i+1}, Bv, t)\right) = \left(M_r(Aw_{2i}, Bv, t)\right)$$
$$\leq \left\{\begin{matrix} a\left(M_r(lw_{2i}, lv, t)\right) + b\left(\frac{M_r(\ell v, lw_{2i+1}, 2t).M_r(lw_{2i}, lv, t).M_r(lv, Bv, t)}{M_r(lw_{2i}, lv, t)}\right) \\ +c\left(M_r(lw_{2i}, lw_{2i+1}, t) + M_r(\ell v, Bv, t)\right) \end{matrix}\right\}$$

$$\left(N_r(lw_{2i+1},Bv,t)\right)=\left(N_r(Aw_{2i},Bv,t)\right)$$
$$\leq\begin{cases}a\left(\mathrm{N_r}(lw_{2i},lv,\mathrm{t})\right)+b\left(\dfrac{\mathrm{N_r}(\ell v,Aw_{2i},2\mathrm{t}).N_r(lw_{2i},Bv,2\mathrm{t})}{\mathrm{N_r}(lw_{2i},lv,\mathrm{t})}\right)\\ +c\left(\mathrm{N_r}(lw_{2i},Aw_{2i},\mathrm{t})+\mathrm{N_r}(\ell v,Bv,\mathrm{t})\right)\end{cases}$$
$$\leq\begin{cases}a\left(\mathrm{N_r}(lw_{2i},lv,\mathrm{t})\right)+b\left(\frac{\mathrm{N_r}(\ell v,lw_{2i+1},2\mathrm{t}).\mathrm{N_r}(lw_{2i},lv,\mathrm{t}).\mathrm{N_r}(lv,Bv,\mathrm{t})}{\mathrm{N_r}(lw_{2i},lv,\mathrm{t})}\right)\\ +c\left(\mathrm{N_r}(lw_{2i},lw_{2i+1},\mathrm{t})+\mathrm{N_r}(\ell v,Bv,\mathrm{t})\right)\end{cases}$$
$$\left(R_r(lw_{2i+1},Bv,t)\right)=\left(R_r(Aw_{2i},Bv,t)\right)$$
$$\leq\begin{cases}a\left(\mathrm{M_r}(lw_{2i},lv,\mathrm{t})\right)+b\left(\frac{\mathrm{M_r}(\ell v,lw_{2i+1},2\mathrm{t}).\mathrm{M_r}(lw_{2i},lv,\mathrm{t}).\mathrm{M_r}(lv,Bv,\mathrm{t})}{\mathrm{M_r}(lw_{2i},lv,\mathrm{t})}\right)\\ +c\left(\mathrm{M_r}(lw_{2i},lw_{2i+1},\mathrm{t})+\mathrm{M_r}(\ell v,Bv,\mathrm{t})\right)\end{cases} \quad 23$$

Then,

$\lim_{i\to\infty}\sup\left(\mathrm{M_r}(\ell w_{2i+1},Bv,\mathrm{t})\right)\leq(b+c)\left(\mathrm{M_r}(\ell v,Av,\mathrm{t})\right)$,
$\lim_{i\to\infty}\sup\left(N_\mathrm{r}(\ell w_{2i+1},Bv,\mathrm{t})\right)\leq(b+c)\left(\mathrm{N_r}(\ell v,Av,\mathrm{t})\right)$ and
$\lim_{i\to\infty}\sup\left(\mathrm{R_r}(\ell w_{2i+1},Bv,\mathrm{t})\right)\leq(b+c)\left(\mathrm{R_r}(\ell v,Av,\mathrm{t})\right)\, for\ t>0.$ 24

From (17), (22), and (24), we get
$\mathrm{M_r}(\ell v,Bv,\mathrm{t})\leq(b+c)\left(\mathrm{M_r}(\ell v,Bv,\mathrm{t})\right)$,
$N_\mathrm{r}(\ell v,Bv,\mathrm{t})\leq(b+c)\left(\mathrm{N_r}(\ell v,Bv,\mathrm{t})\right)$, and
$\mathrm{R_r}(\ell v,Bv,\mathrm{t})\leq(b+c)\left(\mathrm{R_r}(\ell v,Bv,\mathrm{t})\right)\, for\ t>0.$ 25

Note that $(b+c)<1, where\ (a+b+2c)<1$, therefore,
$\mathrm{M_r}\ (\ell v,Bv,t)=\mathrm{M_r}\ (u,Bv,t)=1\Rightarrow u=lv=Av$ for $t>0$.
$\mathrm{N_r}\ (\ell v,Bv,t)=\mathrm{N_r}\ (u,Bv,t)=0\Rightarrow u=lv=Av$ for $t>0$ and
$\mathrm{R_r}\ (\ell v,Bv,t)=\mathrm{R_r}\ (u,Bv,t)=0\Rightarrow u=lv=Av$ for $t>0$.

Hence, u is a CCP of the mappings $\ell,A\ and\ B$ in W such that $u=\ell v=Av=Bv$.

Next, we prove the uniqueness of a coincidence point in $(\mathfrak{H},\mathrm{M_r},N_\mathrm{r},R_\mathrm{r},*,\oplus,\odot)$ for the mappings ℓ,A and B.

Let u^* be the other common coincidence point in Wsuch that $u^*=lv^*=Av^*=Bv^*$ for some $v^*\in W$ Then, from (6) and from, Definition 2.2 (iv, ix, xiv), for $t>0$,

$$\left(\frac{1}{M_r(u,u^*,t)}-1\right)=\left(\frac{1}{M_r(lv,lv^*,t)}-1\right)=\left(\frac{1}{M_r(Av,Bv^*,t)}-1\right)$$
$$=\begin{cases}a\left(\frac{1}{M_r(u^*,u,2t)}-1\right)+b\left(\frac{1}{\frac{M_r(u,u^*,t).M_r(u,u^*,2t)}{M_r(u,u^*,t)}}-1\right)\\ +c\left(\frac{1}{M_r(u,u,t)}-1+\frac{1}{M_r(u^*,u^*,t)}-1\right)\end{cases}$$
$$\leq(a+b)\left(\frac{1}{M_r(u,u^*,t)}-1\right)$$
$$\left(N_r(u,u^*,t)\right)=\left(N_r(lv,lv^*,t)\right)=\left(N_r(Av,Bv^*,t)\right)$$
$$\leq\begin{cases}a\left(\mathrm{N_r}(lv,lv^*,\mathrm{t})\right)+b\left(\frac{\mathrm{N_r}(lv^*,Av,2\mathrm{t}).\mathrm{N_r}(lv,Bv^*,2\mathrm{t})}{\mathrm{N_r}(lv,lv^*,\mathrm{t})}\right)\\ +c\left(N_\mathrm{r}(lv,Av,\mathrm{t})+\mathrm{N_r}(lv^*,Bv^*,\mathrm{t})\right)\end{cases}$$
$$=\begin{cases}a\left(N_r(u^*,u,2t)\right)+b\left(\frac{N_r(u,u^*,t).N_r(u,u^*,2t)}{M_r(u,u^*,t)}\right)\\ +c\left(N_r(u,u,t)+N_r(u^*,u^*,t)\right)\end{cases}$$
$$\leq(a+b)\left(N_r(u,u^*,t)\right)$$
$$\left(R_r(u,u^*,t)\right)=\left(R_r(lv,lv^*,t)\right)=\left(R_r(Av,Bv^*,t)\right)$$
$$\leq\begin{cases}a\left(\mathrm{R_r}(lv,lv^*,\mathrm{t})\right)+b\left(\frac{\mathrm{R_r}(lv^*,Av,2\mathrm{t}).\mathrm{R_r}(lv,Bv^*,2\mathrm{t})}{\mathrm{M_r}(lv,lv^*,\mathrm{t})}\right)\\ +c\left(\mathrm{R_r}(lv,Av,\mathrm{t})+R_\mathrm{r}(lv^*,Bv^*,\mathrm{t})\right)\end{cases}$$
$$=\begin{cases}a\left(R_r(u^*,u,2t)\right)+b\left(\frac{R_r(u,u^*,t).R_r(u,u^*,2t)}{M_r(u,u^*,t)}\right)\\ +c\left(R_r(u,u,t)+R_r(u^*,u^*,t)\right)\end{cases}$$
$$\leq(a+b)\left(R_r(u,u^*,t)\right) \quad 27$$

Note that $(a+b)<1$, where $(a+b+2c)<1$. Thus, we get that $M_r(u,u^*,t)=1$, $N_r(u,u^*,t)=0, R_r(u,u^*,t)=0$ that is, $u=u^*$. By using the weak compatibility of the pair $(A,\ell),(B,\ell)$ and by using Preposition 2.9. We can get a unique CFP of the mappings A,B, and ℓ. Let $\exists\ z\in W$such that, $\ell z=Az=Bz=z$. Hence, we get that
$M_r(v,z,t)=1\Rightarrow v=z, N_r(v,z,t)=0\Rightarrow v=z$,
$R_r(v,z,t)=0\Rightarrow v=z, for\ t>0$.

Corollary 3.2.

Let a NS-sp $(\mathrm{M_r}, N_r, R_r)$ is triangular in a CNM-spaces $(\mathfrak{H}, \mathrm{M_r}, N_r, R_r, *, \oplus, \odot)$ and let $A, B, l \longrightarrow W$ be three self-mappings, satisfies for all $w, x \in W$,

$$\frac{1}{M_r(Aw,bx,t)} - 1 \leq a\left(\frac{1}{M_r(lw,lx,t)} - 1\right) + b\left(\frac{1}{\frac{M_r(lx,Aw,2t).M_r(lw,Bx,2t)}{M_r(lw,lx,t)}} - 1\right),$$

$$N_r(Aw,bx,t) \leq a(N_r(lw,lx,t)) + b\left(\frac{N_r(lx,Aw,2t).N_r(lw,Bx,2t)}{N_r(lw,lx,t)}\right) \text{ and}$$

$$R_r(Aw,bx,t) \leq a(R_r(lw,lx,t)) + b\left(\frac{R_r(lx,Aw,2t).R_r(lw,Bx,2t)}{R_r(lw,lx,t)}\right), \qquad 27$$

for $t > 0$ and $0 \leq a, b < 1$ with $(a + b) < 1$. If $A(W) \cup B(W) \subset \ell(W)$, where $\ell(W)$ is a complete subspace of W. Then, A, B, and ℓ have a unique point of coincidence. Moreover, if the pairs (A, ℓ) and (B, ℓ) are weakly compatible. Then, A, B, and ℓ have a unique CFP in W.

If we use identity map instead of $\ell, i.e., \ell = I$, in Theorem 3.1, we can get the following corollary:

Corollary 3.3.

Let a NM-spaces, $(\mathrm{M_r}, N_r, R_r)$ is triangular in a CNM-space $(\mathfrak{H}, \mathrm{M_r}, N_r, R_r, *, \oplus, \odot)$ and let $A, B: W \longrightarrow W$ be two self-mappings, satisfies for all $w, x \in W$,

$$\frac{1}{M_r(Aw,bx,t)} - 1 \leq \begin{Bmatrix} a\left(\frac{1}{\mathrm{M_r}(w,x,\mathrm{t})} - 1\right) + b\left(\frac{\mathrm{M_r}(w,x,\mathrm{t})}{\mathrm{M_r}(x,Aw,2\mathrm{t}).\mathrm{M_r}(w,Bx,2\mathrm{t})} - 1\right) \\ +c\left(\frac{1}{\mathrm{M_r}(w,Aw,\mathrm{t})} - 1 + \frac{1}{\mathrm{M_r}(x,Bx,\mathrm{t})} - 1\right) \end{Bmatrix}$$

$$N_r(Aw,bx,t) \leq \begin{Bmatrix} a\left(N_\mathrm{r}(w,x,\mathrm{t})\right) + b\left(\frac{\mathrm{N_r}(w,x,\mathrm{t})}{\mathrm{N_r}(x,Aw,2\mathrm{t}).\mathrm{N_r}(w,Bx,2\mathrm{t})}\right) \\ +c\left(N_\mathrm{r}(w,Aw,\mathrm{t}) + \mathrm{N_r}(x,Bx,\mathrm{t})\right) \end{Bmatrix}$$

$$R_r(Aw,bx,t) \leq \begin{Bmatrix} a\left(\mathrm{R_r}(w,x,\mathrm{t})\right) + b\left(\frac{R_\mathrm{r}(w,x,\mathrm{t})}{\mathrm{R_r}(x,Aw,2\mathrm{t}).R_\mathrm{r}(w,Bx,2\mathrm{t})}\right) \\ +c\left(\mathrm{R_r}(w,Aw,\mathrm{t}) + \mathrm{R_r}(x,Bx,\mathrm{t})\right) \end{Bmatrix} \qquad 28$$

$for\ t > 0, 0 \leq a, b, c < 1$ with $(a + b + 2c) < 1$. Then, the mappings A and B have a unique CFP in W.

Example 3.4.

Let $\mathfrak{H} = [0,1]$, $*$ is a product continuous t-norm and $\oplus$ is a product continuous t-conorm on $\mathfrak{H} = [0,1]$ which is defined as $\xi * \zeta = \xi\zeta$, $\xi * \zeta = \xi + \zeta - \xi\zeta$ for all $\xi, \zeta \in \mathfrak{H}$ and a NM-space $M_r, N_r, R_r : \mathfrak{H}^2 \times (0,\infty) \longrightarrow [0,1]$ is defined by

$$M_r(w,x,t) = \frac{t}{t+|w-x|}, N_r(w,x,t) = \frac{|w-x|}{t+|w-x|}, \forall w, x \in \mathfrak{H}, \text{and } t > 0. \qquad 29$$

Then, it is easy to prove that $\mathrm{M_r}, N_r, R_r$ is triangular and $(\mathfrak{H}, \mathrm{M_r}, N_r, R_r, *, \oplus, \odot)$ is a CNM-space. The mappings A, B, ℓ : $\mathfrak{H} \longrightarrow \mathfrak{H}$ be defined as

$$Aw = Bw = \frac{4w}{3w+9} \text{ and } lw = \frac{2w}{3} \forall\ w \in \mathfrak{H}. \qquad 30$$

Then, from (29), we get

$$\frac{1}{M_r(Aw,bx,t)} - 1 = \frac{1}{t}|Aw - Bx| = \frac{1}{t}\left|\frac{4w}{3w+9} - \frac{4x}{3x+9}\right|$$
$$= \left|\frac{36w-36x}{(3w+9)(3x+9)}\right| \leq \left|\frac{36w-36x}{81}\right| = \frac{2}{3}\left|\frac{lw-lx}{t}\right| = \frac{2}{3}\left(\frac{1}{M_r(lw,lx,t)} - 1\right) \text{ for } t > 0.$$
$$N_r(Aw,bx,t) = \frac{1}{t}|Aw - Bx| = \frac{1}{t}\left|\frac{4w}{3w+9} - \frac{4x}{3x+9}\right|$$
$$= \left|\frac{36w-36x}{(3w+9)(3x+9)}\right| \leq \left|\frac{36w-36x}{81}\right| = \frac{2}{3}\left|\frac{lw-lx}{t}\right| = \frac{2}{3}\left(N_r(lw,lx,t)\right) \text{for } t > 0.$$
$$R_r(Aw,bx,t) = \frac{1}{t}|Aw - Bx| = \frac{1}{t}\left|\frac{4w}{3w+9} - \frac{4x}{3x+9}\right|$$
$$= \left|\frac{36w-36x}{(3w+9)(3x+9)}\right| \leq \left|\frac{36w-36x}{81}\right| = \frac{2}{3}\left|\frac{lw-lx}{t}\right| = \frac{2}{3}\left(R_r(lw,lx,t)\right) \text{ for } t > 0. \qquad 31$$

Hence, the self-mappings A, B, and ℓ are satisfied the WNCC in NM-spaces. Next, we simplify the second term of (6), then, from, Definition 2.2 (iv, ix, xiv), and from (29), $for\ t > 0$, we have

$$\frac{1}{\frac{M_r(lx,Aw,2\mathrm{t}).M_r(lw,Bx,2\mathrm{t})}{M_r(lw,lx,\mathrm{t})}} - 1 \leq \frac{1}{\frac{M_r(lx,lw,\mathrm{t}).M_r(lw,Aw,\mathrm{t}).M_r(lw,lx,\mathrm{t}).M_r(lx,Bx,\mathrm{t})}{M_r(lw,lx,\mathrm{t})}} - 1$$
$$= \frac{1}{M_r(lx,lw,\mathrm{t}).M_r(lw,Aw,\mathrm{t}).M_r(lx,Bx,\mathrm{t})} - 1$$
$$\geq \frac{t^3}{(t+|lw-lx|).(t+|lw-Aw|).(t+|lx-Bx|)}$$
$$= \frac{t^3}{(t+|2w/3-2x/3|).(t+|2w/3-4w/3w+9|).(t+|2x/3-4w/3w+9|)}$$

$$= \frac{t^3}{\big((t+|2w/3-2x/3|).(t+|2(w^2+w)/3w+9|).(t+|2(x^2+x)/3x+9|)\big)}$$

$$= \frac{t^3}{\begin{pmatrix} \left(t+\left|\frac{2w}{3}-\frac{2x}{3}\right|\right).(t^2+2t((w^2+w)/3w+9) \\ +\left(x^2+\frac{x}{3x}+9\right) \\ +(4(w^2+w).(x^2+x)/(3w+9).(3x+9) \end{pmatrix}}$$

$$\geq \frac{t^3}{\begin{pmatrix} \left(t+\frac{2}{3|w-x|}\right).(t^2+(2t/81)(3w^2x+3wx^2+9(w^2+x^2)+6wx \\ +9(w+x)+4/81(w^2x^2+w^2x+wx^2+wx)) \end{pmatrix}}$$

$$\leq \frac{\left(t+\frac{2}{3|w-x|}\right).\begin{pmatrix} t^2+2/81(9t(w^2+x^2)+2w^2x^2 \\ +wx(6t+2)+(wx(3t+2)+9(w+x)) \end{pmatrix}}{t^3}$$

$$= \frac{1}{t^3}\begin{pmatrix} \frac{2t^2}{3}|w-x|+ \\ \frac{2}{81}\left(t+\frac{2}{3}|w-x|\right)\begin{pmatrix} 9t(w^2+x^2)+2w^2x^2+wx(6t+2) \\ +(wx(3t+2)+9 \end{pmatrix}(w+x)) \end{pmatrix}$$

$$\frac{N_r(lx,Aw,2\mathrm{t}).N_r(lw,Bx,2\mathrm{t})}{N_r(lw,lx,\mathrm{t})} \leq \frac{N_r(lx,lw,\mathrm{t}).N_r(lw,Aw,\mathrm{t}).N_r(lw,lx,\mathrm{t}).N_r(lx,Bx,\mathrm{t})}{N_r(lw,lx,\mathrm{t})}$$

$$= N_r(lx,lw,\mathrm{t}).N_r(lw,Aw,\mathrm{t}).N_r(lx,Bx,\mathrm{t})$$

$$\geq \frac{t^3}{(t+|lw-lx|).(t+|lw-Aw|).(t+|lx-Bx|)}$$

$$= \frac{t^3}{(t+|2w/3-2x/3|).(t+|2w/3-4w/3w+9|).(t+|2x/3-4w/3w+9|)}$$

$$= \frac{t^3}{\big((t+|2w/3-2x/3|).(t+|2(w^2+w)/3w+9|).(t+|2(x^2+x)/3x+9|)\big)}$$

$$= \frac{t^3}{\begin{pmatrix} \left(t+\left|\frac{2w}{3}-\frac{2x}{3}\right|\right).(t^2+2t((w^2+w)/3w+9)+\left(x^2+\frac{x}{3x}+9\right) \\ +(4(w^2+w).(x^2+x)/(3w+9).(3x+9) \end{pmatrix}}$$

$$\geq \frac{t^3}{\begin{pmatrix} \left(t+\frac{2}{3|w-x|}\right).(t^2+(2t/81)(3w^2x+3wx^2 \\ +9(w^2+x^2)+6wx+9(w+x)+4/81(w^2x^2+w^2x+wx^2+wx)) \end{pmatrix}}$$

$$\leq \frac{\left(t+\frac{2}{3|w-x|}\right).\begin{pmatrix} t^2+2/81(9t\left(w^2+x^2\right) \\ +2w^2x^2+wx(6t+2)+(wx(3t+2)+9(w+x)) \end{pmatrix}}{t^3}$$

$$= \frac{1}{t^3}\left(\frac{2t^2}{3}|w-x|+\frac{2}{81}\left(t+\frac{2}{3}|w-x|\right)\begin{pmatrix} 9t(w^2+x^2)+2w^2x^2 \\ +wx(6t+2) \\ +(wx(3t+2)+9 \end{pmatrix}(w+x)\right)$$

$$\frac{R_r(lx,Aw,2\mathrm{t}).R_r(lw,Bx,2\mathrm{t})}{R_r(lw,lx,\mathrm{t})} \leq \frac{R_r(lx,lw,\mathrm{t}).R_r(lw,Aw,\mathrm{t}).R_r(lw,lx,\mathrm{t}).R_r(lx,Bx,\mathrm{t})}{R_r(lw,lx,\mathrm{t})}$$

$$= R_r(lx,lw,\mathrm{t}).R_r(lw,Aw,\mathrm{t}).R_r(lx,Bx,\mathrm{t})$$

$$\geq \frac{t^3}{(t+|lw-lx|).(t+|lw-Aw|).(t+|lx-Bx|)}$$

$$= \frac{t^3}{(t+|2w/3-2x/3|).(t+|2w/3-4w/3w+9|).(t+|2x/3-4w/3w+9|)}$$

$$= \frac{t^3}{\big((t+|2w/3-2x/3|).(t+|2(w^2+w)/3w+9|).(t+|2(x^2+x)/3x+9|)\big)}$$

$$= \frac{t^3}{\begin{pmatrix} \left(t+\left|\frac{2w}{3}-\frac{2x}{3}\right|\right).(t^2+2t((w^2+w)/3w+9) \\ +\left(x^2+\frac{x}{3x}+9\right) \\ +(4(w^2+w).(x^2+x)/(3w+9).(3x+9) \end{pmatrix}}$$

$$\geq \frac{t^3}{\begin{pmatrix}\left(t+\frac{2}{3|w-x|}\right).(t^2+(2t/81)(3w^2x+3wx^2\\+9(w^2+x^2)+6wx\\+9(w+x)+4/81(w^2x^2+w^2x+wx^2+wx))\end{pmatrix}}$$

$$\leq \frac{\left(t+\frac{2}{3|w-x|}\right).\begin{pmatrix}t^2+2/81(9t(w^2+x^2)\\+2w^2x^2+wx(6t+2)+(wx(3t+2)+9(w+x))\end{pmatrix}}{t^3}$$

$$=\frac{1}{t^3}\left(\frac{2t^2}{3}|w-x|+\frac{2}{81}\left(t+\frac{2}{3}|w-x|\right)\begin{pmatrix}9t(w^2+x^2)\\+2w^2x^2+wx(6t+2)\\+(wx(3t+2)+9\end{pmatrix}(w+x))\right) \quad 3.32$$

Lastly, we simplify the third term of (6), then from (29), for $t > 0$

$$\frac{1}{M_r(w,Aw,t)}-1+\frac{1}{M_r(x,Bx,t)}-1=\frac{1}{t}(|lw-Aw|+|lx-Bx|)$$
$$=\frac{1}{t}\left(\left|\frac{2w}{3}-\frac{4w}{3w+9}\right|+\left|\frac{2x}{3}-\frac{4x}{3x+9}\right|\right)=\frac{1}{t}\left(\frac{2(w+w^2)}{3w+9}+\frac{2(x+x^2)}{3x+9}\right)$$
$$\leq\frac{2}{81t}\left((3wx+9)(w+x)+9(\mathrm{w}^2+\mathrm{x}^2)+6\mathrm{wx}\right)$$

$$N_r(w,Aw,t)+N_r(x,Bx,t)=\frac{1}{t}(|lw-Aw|+|lx-Bx|)$$
$$=\frac{1}{t}\left(\left|\frac{2w}{3}-\frac{4w}{3w+9}\right|+\left|\frac{2x}{3}-\frac{4x}{3x+9}\right|\right) \quad 33$$
$$=\frac{1}{t}\left(\frac{2(w+w^2)}{3w+9}+\frac{2(x+x^2)}{3x+9}\right)$$
$$\leq\frac{2}{81t}\left((3wx+9)(w+x)+9(\mathrm{w}^2+\mathrm{x}^2)+6\mathrm{wx}\right)$$

$$R_r(w,Aw,t)+R_r(x,Bx,t)=\frac{1}{t}(|lw-Aw|+|lx-Bx|)$$
$$=\frac{1}{t}\left(\left|\frac{2w}{3}-\frac{4w}{3w+9}\right|+\left|\frac{2x}{3}-\frac{4x}{3x+9}\right|\right)=\frac{1}{t}\left(\frac{2(w+w^2)}{3w+9}+\frac{2(x+x^2)}{3x+9}\right)$$
$$\leq\frac{2}{81t}\left((3wx+9)(w+x)+9(\mathrm{w}^2+\mathrm{x}^2)+6\mathrm{wx}\right)$$

Hence, all the conditions of Theorem 3.1 are satisfied with $a=\frac{2}{3}, b=\frac{1}{9}$, and $c=\frac{1}{5}$. The mappings $A, B, and\ \ell$ have a unique CFP, that is, 0.

Theorem 3.5.

Let a NM-space, $(\mathrm{M_r}, N_\mathrm{r}, R_\mathrm{r})$ be triangular in a CNM-space $(\mathfrak{H}, \mathrm{M_r}, N_\mathrm{r}, R_\mathrm{r}, *, \oplus, \odot)$ and let $A, B, \ell : W \longrightarrow W$ be three self-mappings, satisfies for all $w, x \in W$,

$$\frac{1}{M_r(Aw,Bx,t)}-1\leq\left\{\begin{matrix}a\left(\frac{1}{M_r(lw,lx,t)}-1\right)\\+b\left(\frac{1}{U(A,B,l,w,x,t)}-1\right)+c\left(\frac{1}{\frac{M_r(lw,Bx,2t).M_r(lw,lx,t).M_r(lx,Aw,t)}{M_r(lw,Aw,t).M_r(lx,Bx,t)}}-1\right)\end{matrix}\right\},$$

$$N_r(Aw,Bx,t)\leq\left\{\begin{matrix}a\left(N_r(lw,lx,t)\right)\\+b(U(A,B,l,w,x,t))+c\left(\frac{N_r(lw,Bx,2t).N_r(lw,lx,t).N_r(lx,Aw,t)}{N_r(lw,Aw,t).N_r(lx,Bx,t)}\right)\end{matrix}\right\},$$

$$R_r(Aw,Bx,t)\leq\left\{\begin{matrix}a\left(R_r(lw,lx,t)\right)\\+b(U(A,B,l,w,x,t))+c\left(\frac{R_r(lw,Bx,2t).R_r(lw,lx,t).R_r(lx,Aw,t)}{R_r(lw,Aw,t).R_r(lx,Bx,t)}\right)\end{matrix}\right\}, \quad 34$$

Where,

$$\mathbf{U}(A,B,\ell,w,x,t)=\left\{\begin{matrix}\frac{1}{M_r(lw,lx,t)}-1,\frac{1}{M_r(lw,Aw,t)}-1,\\\frac{1}{M_r(lx,Bx,t)M_r(lx,Aw,t)M_r(lw,Bx,t)}-1\end{matrix}\right\}$$

$$\mathbf{U}(A,B,\ell,w,x,t)=\left\{\begin{matrix}N_r(lw,lx,t),N_r(lw,Aw,t),\\N_r(lx,Bx,t)N_r(lx,Aw,t)N_r(lw,Bx,t)\end{matrix}\right\}$$

$$\mathbf{U}(A,B,\ell,w,x,t)=\left\{\begin{matrix}R_r(lw,lx,t),R_r(lw,Aw,t),\\R_r(lx,Bx,t)R_r(lx,Aw,t)R_r(lw,Bx,t)\end{matrix}\right\} \quad 35$$

for $t > 0$ and $0 \leq a, b, c < 1$ with $(a + b + c) < 1$.

If $A(W) \cup B(W) \subset \ell(W)$, where $\ell(W)$ is a complete subspace of W. Then $A, B,$ and ℓ have a coincidence point in W.

Proof.

Let $w_0$be the arbitrary point of W. By the condition $A(W) \cup B(W) \subset \ell(W)$ choose a sequence (w_i) in Wsuch that

$\ell w_{2i+1} = Aw_{2i}$ *and* $\ell w_{2i+2} = Bw_{2i+1}$, for all $i \geq 0$. 36

Now, by (34), for $t > 0$,

$$M_r(\ell w_{2i+1}, \ell w_{2i+2}, t) = M_r(Aw_{2i}, Bw_{2i+1}, t)$$
$$\leq \left\{ \begin{array}{c} a\big(M_r(\ell w_{2i}, \ell w_{2i+1}, t)\big) + b\big(U(A, B, l, w_{2i}, w_{2i+1}, t)\big) \\ +c\left(\dfrac{M_r(\ell w_{2i}, Bw_{2i+1}, 2t). M_r(\ell w_{2i}, \ell w_{2i+1}, t). M_r(\ell w_{2i+1}, Aw_{2i}, 2t)}{M_r(\ell w_{2i}, Aw_{2i}, t). M_r(\ell w_{2i+1}, Bw_{2i+1}, t)}\right) \end{array} \right\}$$
$$= \left\{ \begin{array}{c} a\big(M_r(\ell w_{2i}, \ell w_{2i+1}, t)\big) + b\big(U(A, B, l, w_{2i}, w_{2i+1}, t)\big) \\ +c\left(\frac{M_r(\ell w_{2i}, lw_{2i+2}, 2t). M_r(\ell w_{2i}, \ell w_{2i+1}, t)}{M_r(\ell w_{2i}, lw_{2i+1}, t). M_r(\ell w_{2i+1}, lw_{2i+2}, t)}\right) \end{array} \right\}$$

$$N_r(\ell w_{2i+1}, \ell w_{2i+2}, t) = N_r(Aw_{2i}, Bw_{2i+1}, t)$$
$$\leq \left\{ \begin{array}{c} a\big(N_r(\ell w_{2i}, \ell w_{2i+1}, t)\big) + b\big(U(A, B, l, w_{2i}, w_{2i+1}, t)\big) \\ +c\left(\dfrac{N_r(\ell w_{2i}, Bw_{2i+1}, 2t). N_r(\ell w_{2i}, \ell w_{2i+1}, t). N_r(\ell w_{2i+1}, Aw_{2i}, 2t)}{N_r(\ell w_{2i}, Aw_{2i}, t). N_r(\ell w_{2i+1}, Bw_{2i+1}, t)}\right) \end{array} \right\}$$
$$= \left\{ \begin{array}{c} a\big(N_r(\ell w_{2i}, \ell w_{2i+1}, t)\big) + b\big(U(A, B, l, w_{2i}, w_{2i+1}, t)\big) \\ +c\left(\frac{N_r(\ell w_{2i}, lw_{2i+2}, 2t). N_r(\ell w_{2i}, \ell w_{2i+1}, t)}{N_r(\ell w_{2i}, lw_{2i+1}, t). N_r(\ell w_{2i+1}, lw_{2i+2}, t)}\right) \end{array} \right\}$$

$$R_r(\ell w_{2i+1}, \ell w_{2i+2}, t) = R_r(Aw_{2i}, Bw_{2i+1}, t)$$
$$\leq \left\{ \begin{array}{c} a\big(R_r(\ell w_{2i}, \ell w_{2i+1}, t)\big) + b\big(U(A, B, l, w_{2i}, w_{2i+1}, t)\big) \\ +c\left(\dfrac{R_r(\ell w_{2i}, Bw_{2i+1}, 2t). R_r(\ell w_{2i}, \ell w_{2i+1}, t). R_r(\ell w_{2i+1}, Aw_{2i}, 2t)}{R_r(\ell w_{2i}, Aw_{2i}, t). R_r(\ell w_{2i+1}, Bw_{2i+1}, t)}\right) \end{array} \right\}$$
$$= \left\{ \begin{array}{c} a\big(R_r(\ell w_{2i}, \ell w_{2i+1}, t)\big) + b\big(U(A, B, l, w_{2i}, w_{2i+1}, t)\big) \\ +c\left(\frac{R_r(\ell w_{2i}, lw_{2i+2}, 2t). R_r(\ell w_{2i}, \ell w_{2i+1}, t)}{R_r(\ell w_{2i}, lw_{2i+1}, t). R_r(\ell w_{2i+1}, lw_{2i+2}, t)}\right) \end{array} \right\} \quad 37$$

Where

$$\boldsymbol{U}(A, B, \ell, w_{2i}, w_{2i+1}, t)$$
$$= min\left\{ \begin{array}{c} M_r(\ell w_{2i}, lw_{2i+1}, t), M_r(\ell w_{2i}, Aw_{2i}, t) \\ M_r(\ell w_{2i+1}, Bw_{2i+1}, t), M_r(\ell w_{2i+1}, Aw_{2i}, t), M_r(\ell w_{2i}, Bw_{2i+1}, t) \end{array} \right\}$$
$$= min\left\{ \begin{array}{c} M_r(\ell w_{2i}, lw_{2i+1}, t), M_r(\ell w_{2i}, lw_{2i+1}, t) \\ M_r(\ell w_{2i+1}, lw_{2i+2}, t), M_r(\ell w_{2i+1}, lw_{2i+2}, t), M_r(\ell w_{2i}, lw_{2i+2}, t) \end{array} \right\}$$
$$= min\left\{ \begin{array}{c} M_r(\ell w_{2i}, lw_{2i+1}, t), M_r(\ell w_{2i+1}, lw_{2i+2}, t), \\ 0, M_r(\ell w_{2i}, lw_{2i+2}, t) \end{array} \right\} = 0.$$

$$\boldsymbol{U}(A, B, \ell, w_{2i}, w_{2i+1}, t)$$
$$= max\left\{ \begin{array}{c} N_r(\ell w_{2i}, lw_{2i+1}, t), N_r(\ell w_{2i}, Aw_{2i}, t) \\ N_r(\ell w_{2i+1}, Bw_{2i+1}, t), N_r(\ell w_{2i+1}, Aw_{2i}, t), N_r(\ell w_{2i}, Bw_{2i+1}, t) \end{array} \right\}$$
$$= max\left\{ \begin{array}{c} N_r(\ell w_{2i}, lw_{2i+1}, t), N_r(\ell w_{2i}, lw_{2i+1}, t) \\ N_r(\ell w_{2i+1}, lw_{2i+2}, t), N_r(\ell w_{2i+1}, lw_{2i+2}, t), N_r(\ell w_{2i}, lw_{2i+2}, t) \end{array} \right\}$$
$$= max\left\{ \begin{array}{c} N_r(\ell w_{2i}, lw_{2i+1}, t), N_r(\ell w_{2i+1}, lw_{2i+2}, t), \\ 0, N_r(\ell w_{2i}, lw_{2i+2}, t) \end{array} \right\} = 0.$$

$$\boldsymbol{U}(A, B, \ell, w_{2i}, w_{2i+1}, t)$$
$$= max\left\{ \begin{array}{c} R_r(\ell w_{2i}, lw_{2i+1}, t), R_r(\ell w_{2i}, Aw_{2i}, t) \\ R_r(\ell w_{2i+1}, Bw_{2i+1}, t), R_r(\ell w_{2i+1}, Aw_{2i}, t), R_r(\ell w_{2i}, Bw_{2i+1}, t) \end{array} \right\}$$
$$= max\left\{ \begin{array}{c} R_r(\ell w_{2i}, lw_{2i+1}, t), R_r(\ell w_{2i}, lw_{2i+1}, t) \\ R_r(\ell w_{2i+1}, lw_{2i+2}, t), R_r(\ell w_{2i+1}, lw_{2i+2}, t), R_r(\ell w_{2i}, lw_{2i+2}, t) \end{array} \right\}$$
$$= max\left\{ \begin{array}{c} R_r(\ell w_{2i}, lw_{2i+1}, t), R_r(\ell w_{2i+1}, lw_{2i+2}, t), \\ 0, R_r(\ell w_{2i}, lw_{2i+2}, t) \end{array} \right\} = 0. \quad 38$$

From (37), (38), and by using Definition 2.2 (iv), for $t > 0$, we obtain

$$\frac{1}{M_r(\ell w_{2i+1}, \ell w_{2i+2}, t)} - 1 \leq \left\{ \begin{array}{c} a\left(\frac{1}{M_r(\ell w_{2i}, \ell w_{2i+1}, t)} - 1\right) \\ +c\left(\frac{1}{\frac{M_r(\ell w_{2i}, lw_{2i+1}, t). M_r(\ell w_{2i+1}, \ell w_{2i+2}, t). M_r(\ell w_{2i}, lw_{2i+1}, t)}{M_r(\ell w_{2i}, lw_{2i+1}, t). M_r(\ell w_{2i+1}, lw_{2i+2}, t)}} - 1\right) \end{array} \right\}$$

$$N_r(\ell w_{2i+1}, \ell w_{2i+2}, t) \leq \left\{ \begin{array}{c} a\big(N_r(\ell w_{2i}, \ell w_{2i+1}, t)\big) \\ +c\left(\frac{N_r(\ell w_{2i}, lw_{2i+1}, t). N_r(\ell w_{2i+1}, \ell w_{2i+2}, t). N_r(\ell w_{2i}, lw_{2i+1}, t)}{N_r(\ell w_{2i}, lw_{2i+1}, t). N_r(\ell w_{2i+1}, lw_{2i+2}, t)}\right) \end{array} \right\}$$

$$R_r(\ell w_{2i+1}, \ell w_{2i+2}, t) \leq \left\{ \begin{array}{c} a\big(R_r(\ell w_{2i}, \ell w_{2i+1}, t)\big) \\ +c\left(\frac{R_r(\ell w_{2i}, lw_{2i+1}, t). R_r(\ell w_{2i+1}, \ell w_{2i+2}, t). R_r(\ell w_{2i}, lw_{2i+1}, t)}{R_r(\ell w_{2i}, lw_{2i+1}, t). R_r(\ell w_{2i+1}, lw_{2i+2}, t)}\right) \end{array} \right\} \quad 39$$

After simplification, for $t > 0$,

$$R_r(\ell w_{2i+2}, \ell w_{2i+3}, t) = R_r(Aw_{2i+2}, Bw_{2i+1}, t)$$
$$\leq \left\{ \begin{array}{c} a\big(R_r(\ell w_{2i+2}, \ell w_{2i+1}, t)\big) \\ +b\big(U(A,B,l,w_{2i+2},w_{2i+1},t)\big) \\ +c\left(\frac{R_r(\ell w_{2i+2}, Bw_{2i+1}, 2t).R_r(\ell w_{2i+2}, \ell w_{2i+1}, t).R_r(\ell w_{2i+1}, Aw_{2i+2}, 2t)}{R_r(\ell w_{2i+2}, Aw_{2i+2}, t).R_r(\ell w_{2i+1}, Bw_{2i+1}, t)}\right) \end{array} \right\}$$
$$= \left\{ \begin{array}{c} a\big(R_r(\ell w_{2i+2}, \ell w_{2i+1}, t)\big) + b\big(U(A,B,l,w_{2i+2},w_{2i+1},t)\big) \\ +c\left(\frac{R_r(\ell w_{2i+2}, lw_{2i+1}, t).R_r(\ell w_{2i+1}, \ell w_{2i+3}, 2t)}{R_r(\ell w_{2i+2}, lw_{2i+3}, t).R_r(\ell w_{2i+1}, lw_{2i+2}, t)}\right) \end{array} \right\} \quad 41$$

Where,

$$\boldsymbol{U}(A,B,\ell,w_{2i+2},w_{2i+1},t) = min\left\{ \begin{array}{c} M_r(\ell w_{2i+2}, lw_{2i+1}, t), M_r(\ell w_{2i+2}, Aw_{2i+2}, t) \\ M_r(\ell w_{2i+1}, Bw_{2i+1}, t), M_r(\ell w_{2i+1}, Aw_{2i+2}, t), \\ M_r(\ell w_{2i+2}, Bw_{2i+1}, t) \end{array} \right\}$$
$$= min\left\{ \begin{array}{c} M_r(\ell w_{2i+2}, lw_{2i+1}, t), M_r(\ell w_{2i+2}, lw_{2i+3}, t) \\ M_r(\ell w_{2i+1}, lw_{2i+2}, t), M_r(\ell w_{2i+2}, lw_{2i+3}, t), \\ M_r(\ell w_{2i+2}, lw_{2i+2}, t) \end{array} \right\}$$
$$= min\{M_r(\ell w_{2i+2}, lw_{2i+1}, t), M_r(\ell w_{2i+2}, lw_{2i+3}, t), M_r(\ell w_{2i+1}, lw_{2i+3}, t), 0\}$$
$$\boldsymbol{U}(A,B,\ell,w_{2i+2},w_{2i+1},t) = max\left\{ \begin{array}{c} N_r(\ell w_{2i+2}, lw_{2i+1}, t), N_r(\ell w_{2i+2}, Aw_{2i+2}, t) \\ N_r(\ell w_{2i+1}, Bw_{2i+1}, t), N_r(\ell w_{2i+1}, Aw_{2i+2}, t), \\ N_r(\ell w_{2i+2}, Bw_{2i+1}, t) \end{array} \right\}$$
$$= max\left\{ \begin{array}{c} N_r(\ell w_{2i+2}, lw_{2i+1}, t), N_r(\ell w_{2i+2}, lw_{2i+3}, t) \\ N_r(\ell w_{2i+1}, lw_{2i+2}, t), N_r(\ell w_{2i+2}, lw_{2i+3}, t), \\ N_r(\ell w_{2i+2}, lw_{2i+2}, t) \end{array} \right\}$$
$$= max\{N_r(\ell w_{2i+2}, lw_{2i+1}, t), N_r(\ell w_{2i+2}, lw_{2i+3}, t), N_r(\ell w_{2i+1}, lw_{2i+3}, t), 0\}$$
$$\boldsymbol{U}(A,B,\ell,w_{2i+2},w_{2i+1},t) = max\left\{ \begin{array}{c} R_r(\ell w_{2i+2}, lw_{2i+1}, t), R_r(\ell w_{2i+2}, Aw_{2i+2}, t) \\ R_r(\ell w_{2i+1}, Bw_{2i+1}, t), R_r(\ell w_{2i+1}, Aw_{2i+2}, t), \\ R_r(\ell w_{2i+2}, Bw_{2i+1}, t) \end{array} \right\}$$
$$= max\left\{ \begin{array}{c} R_r(\ell w_{2i+2}, lw_{2i+1}, t), R_r(\ell w_{2i+2}, lw_{2i+3}, t) \\ R_r(\ell w_{2i+1}, lw_{2i+2}, t), R_r(\ell w_{2i+2}, lw_{2i+3}, t), R_r(\ell w_{2i+2}, lw_{2i+2}, t) \end{array} \right\}$$
$$= max\{R_r(\ell w_{2i+2}, lw_{2i+1}, t), R_r(\ell w_{2i+2}, lw_{2i+3}, t), R_r(\ell w_{2i+1}, lw_{2i+3}, t), 0\} \quad 42$$

Now from (41), (42), and by using Definition 2.2 (iv),

$$\frac{1}{M_r(\ell w_{2i+2}, \ell w_{2i+3}, t)} - 1 \leq \left\{ \begin{array}{c} a\big(M_r(\ell w_{2i+2}, \ell w_{2i+1}, t)\big) \\ +c\left(\frac{M_r(\ell w_{2i+2}, lw_{2i+1}, t).M_r(\ell w_{2i+1}, \ell w_{2i+2}, t).M_r(\ell w_{2i+2}, lw_{2i+3}, t)}{M_r(\ell w_{2i+2}, lw_{2i+3}, t).M_r(\ell w_{2i+1}, lw_{2i+2}, t)}\right) \end{array} \right\}$$
$$N_r(\ell w_{2i+2}, \ell w_{2i+3}, t) \leq \left\{ \begin{array}{c} a\big(N_r(\ell w_{2i+2}, \ell w_{2i+1}, t)\big) \\ +c\left(\frac{N_r(\ell w_{2i+2}, lw_{2i+1}, t).N_r(\ell w_{2i+1}, \ell w_{2i+2}, t).N_r(\ell w_{2i+2}, lw_{2i+3}, t)}{N_r(\ell w_{2i+2}, lw_{2i+3}, t).N_r(\ell w_{2i+1}, lw_{2i+2}, t)}\right) \end{array} \right\}$$
$$R_r(\ell w_{2i+2}, \ell w_{2i+3}, t) \leq \left\{ \begin{array}{c} a\big(R_r(\ell w_{2i+2}, \ell w_{2i+1}, t)\big) \\ +c\left(\frac{R_r(\ell w_{2i+2}, lw_{2i+1}, t).R_r(\ell w_{2i+1}, \ell w_{2i+2}, t).R_r(\ell w_{2i+2}, lw_{2i+3}, t)}{R_r(\ell w_{2i+2}, lw_{2i+3}, t).R_r(\ell w_{2i+1}, lw_{2i+2}, t)}\right) \end{array} \right\} \quad 43$$

Then after simplification, for $t > 0$,

$$\frac{1}{M_r(\ell w_{2i+2}, \ell w_{2i+3}, t)} - 1 \leq \zeta\left(\frac{1}{M_r(\ell w_{2i+1}, \ell w_{2i+2}, t)} - 1\right) \text{ where } \zeta = (a + c) < 1$$
$$N_r(\ell w_{2i+2}, \ell w_{2i+3}, t) \leq \zeta\big(N_r(\ell w_{2i+1}, \ell w_{2i+2}, t)\big) \text{ where } \zeta = (a + c) < 1$$
$$R_r(\ell w_{2i+2}, \ell w_{2i+3}, t) \leq \zeta\big(R_r(\ell w_{2i+1}, \ell w_{2i+2}, t)\big) \text{ where } \zeta = (a + c) < 1 \quad 44$$

Now, from (40), (44), and by induction,

$$R_r(\ell w_{2i+2}, \ell w_{2i+3}, t) \leq \zeta\big(R_r(lw_{2i+1}, lw_{2i+2}, t)\big)$$
$$\leq \zeta^2\big(R_r(\ell w_{2i}, \ell w_{2i+1}, t)\big)$$
$$\leq \cdots \leq \zeta^{2i+2}\big(R_r(\ell w_0, \ell w_1, t)\big) \longrightarrow 0 \text{ as } i \longrightarrow \infty. \quad 45$$

Hence, (ℓw_i) is a neutrosophic-contractive sequence in $(\mathfrak{H}, \mathrm{M_r}, N_r, R_r, *, \oplus, \odot)$, therefore,

$$\lim_{i\to\infty} M_r(\ell w_i, \ell w_{i-1}, t) = 1 \; for \; t > 0, \; \lim_{i\to\infty} N_r(\ell w_i, \ell w_{i-1}, t) = 0 \; for \; t > 0 \text{ and}$$
$$\lim_{i\to\infty} R_r(\ell w_i, \ell w_{i-1}, t) = 0 \; for \; t > 0 \quad 46$$

Since M_r, N_r, R_r is triangular, $j > i > i_0$,

$$\frac{1}{M_r\big(\ell w_i, \ell w_j, t\big)} - 1 \leq \left\{ \begin{array}{c} M_r(lw_i, lw_{i+1}, t) + M_r(lw_{i+1}, lw_{i+2}, t) \\ + \cdots + M_r\big(lw_{i-1}, lw_j, t\big) \end{array} \right\}$$

$$\leq \begin{Bmatrix} \zeta^i\big(M_r(\ell w_i,\ell w_{i+1},t)\big)+\zeta^{i+1}\big(M_r(\ell w_{i+1},\ell w_{i+2},t)\big) \\ +\cdots+\zeta^{j-1}\left(M_r\big(\ell w_{i-1},\ell w_j,t\big)\right) \end{Bmatrix}$$

$$\leq (\zeta^i+\zeta^{i+1}+\cdots+\zeta^{j-1})\big(M_r(\ell w_0,\ell w_1,t)\big)$$

$$\leq \left(\frac{\zeta^i}{1-\zeta}\right)\big(M_r(\ell w_0,\ell w_1,t)\big)\longrightarrow 0\ as\ \ i\longrightarrow\infty$$

$$N_r\big(\ell w_i,\ell w_j,t\big)\leq \begin{Bmatrix} N_r(lw_i,lw_{i+1},t)+N_r(lw_{i+1},lw_{i+2},t) \\ +\cdots+N_r\big(lw_{i-1},lw_j,t\big) \end{Bmatrix}$$

$$\leq \begin{Bmatrix} \zeta^i\big(N_r(\ell w_i,\ell w_{i+1},t)\big)+\zeta^{i+1}\big(N_r(\ell w_{i+1},\ell w_{i+2},t)\big) \\ +\cdots+\zeta^{j-1}\left(N_r\big(\ell w_{i-1},\ell w_j,t\big)\right) \end{Bmatrix}$$

$$\leq (\zeta^i+\zeta^{i+1}+\cdots+\zeta^{j-1})\big(N_r(\ell w_0,\ell w_1,t)\big)$$

$$\leq \left(\frac{\zeta^i}{1-\zeta}\right)\big(N_r(\ell w_0,\ell w_1,t)\big)\longrightarrow 0\ as\ \ i\longrightarrow\infty.$$

$$R_r\big(\ell w_i,\ell w_j,t\big)\leq \begin{Bmatrix} R_r(lw_i,lw_{i+1},t)+R_r(lw_{i+1},lw_{i+2},t) \\ +\cdots+R_r\big(lw_{i-1},lw_j,t\big) \end{Bmatrix}$$

$$\leq \begin{Bmatrix} \zeta^i\big(R_r(\ell w_i,\ell w_{i+1},t)\big)+\zeta^{i+1}\big(R_r(\ell w_{i+1},\ell w_{i+2},t)\big) \\ +\cdots+\zeta^{j-1}\left(R_r\big(\ell w_{i-1},\ell w_j,t\big)\right) \end{Bmatrix}$$

$$\leq (\zeta^i+\zeta^{i+1}+\cdots+\zeta^{j-1})\big(R_r(\ell w_0,\ell w_1,t)\big)$$

$$\leq \left(\frac{\zeta^i}{1-\zeta}\right)\big(R_r(\ell w_0,\ell w_1,t)\big)\longrightarrow 0\ as\ \ i\longrightarrow\infty. \qquad 47$$

This shows that (ℓw_i) is a Cauchy sequence and $\ell(W)$ is a complete subspace of W. Hence, $\exists\, u,v\ \in W$ such that $\ell w_i \longrightarrow u\ =\ \ell v\ as\ i\longrightarrow\infty, i.e.$

$\lim_{i\to\infty} M_r(u,\ell w_i,t)=M_r(u,\ell v,t)=1\ for\ t>0,$

$\lim_{i\to\infty} N_r(u,\ell w_i,t)=N_r(u,\ell v,t)=0\ for\ t>0$ and

$\lim_{i\to\infty} R_r(u,\ell w_i,t)=R_r(u,\ell v,t)=0\ for\ t>0.$ 48

Since M_r, N_r, R_r is triangular,

$\frac{1}{M_r(\ell v,Av,t)}-1\leq \left(\frac{1}{M_r(lv,lw_{2i+2},t)}-1\right)+\left(\frac{1}{M_r(lw_{2i+2},Av,t)}-1\right)\ for\ t>0,$

$N_r(\ell v,Av,t)\leq \big(N_r(lv,lw_{2i+2},t)\big)+\big(N_r(lw_{2i+2},Av,t)\big)\ for\ t>0,$

$R_r(\ell v,Av,t)\leq \big(R_r(lv,lw_{2i+2},t)\big)+\big(R_r(lw_{2i+2},Av,t)\big)\ for\ t>0.$ 3.49

Now, from (34), (46), (48), and by using Definition 2.2 (iv), for $t>0$, we have that

$$\frac{1}{M_r(\ell w_{2i+2},Av,t)}-1=\frac{1}{M_r(Av,Bw_{2i+1},t)}-1$$

$$\leq \begin{Bmatrix} a\big(M_r(\ell v,\ell w_{2i+1},t)\big)+b\big(U(A,B,l,v,w_{2i+1},t)\big) \\ +c\left(\frac{M_r(\ell v,lw_{2i+2},t).M_r(\ell v,\ell w_{2i+1},t).M_r(\ell w_{2i+1},\ell v,t).M_r(\ell v,Av,t)}{M_r(\ell v,Av,t).M_r(\ell w_{2i+1},lw_{2i+2},t)}\right) \end{Bmatrix}$$

$$\to b(U(A,B,l,v,w_{2i+1},t))\ as\ t\to\infty.$$

$$N_r(\ell w_{2i+2},Av,t)=N_r(Av,Bw_{2i+1},t)$$

$$\leq \begin{Bmatrix} a\big(N_r(\ell v,\ell w_{2i+1},t)\big)+b\big(U(A,B,l,v,w_{2i+1},t)\big) \\ +c\left(\frac{N_r(\ell v,Bw_{2i+1},2t).N_r(\ell v,\ell w_{2i+1},t).N_r(\ell w_{2i+1},Av,2t)}{N_r(\ell v,Av,t).N_r(\ell w_{2i+1},Bw_{2i+1},t)}\right) \end{Bmatrix}$$

$$\leq \begin{Bmatrix} a\big(N_r(\ell v,\ell w_{2i+1},t)\big)+b\big(U(A,B,l,v,w_{2i+1},t)\big) \\ +c\left(\frac{N_r(\ell v,lw_{2i+2},t).N_r(\ell v,\ell w_{2i+1},t).N_r(\ell w_{2i+1},\ell v,t).N_r(\ell v,Av,t)}{N_r(\ell v,Av,t).N_r(\ell w_{2i+1},lw_{2i+2},t)}\right) \end{Bmatrix}$$

$$\to b(U(A,B,l,v,w_{2i+1},t))\ as\ t\to\infty.$$

$$R_r(\ell w_{2i+2},Av,t)=R_r(Av,Bw_{2i+1},t)$$

$$\leq \begin{Bmatrix} a\big(R_r(\ell v,\ell w_{2i+1},t)\big)+b\big(U(A,B,l,v,w_{2i+1},t)\big) \\ +c\left(\frac{R_r(\ell v,Bw_{2i+1},2t).R_r(\ell v,\ell w_{2i+1},t).R_r(\ell w_{2i+1},Av,2t)}{R_r(\ell v,Av,t).R_r(\ell w_{2i+1},Bw_{2i+1},t)}\right) \end{Bmatrix}$$

$$\leq \begin{Bmatrix} a\big(R_r(\ell v,\ell w_{2i+1},t)\big)+b\big(U(A,B,l,v,w_{2i+1},t)\big) \\ +c\left(\frac{R_r(\ell v,lw_{2i+2},t).R_r(\ell v,\ell w_{2i+1},t).R_r(\ell w_{2i+1},\ell v,t).R_r(\ell v,Av,t)}{R_r(\ell v,Av,t).R_r(\ell w_{2i+1},lw_{2i+2},t)}\right) \end{Bmatrix}$$

$$\to b(U(A,B,l,v,w_{2i+1},t))\ as\ t\to\infty. \qquad 50$$

Where,

$$U(A,B,l,v,w_{2i+1},t)=min\begin{Bmatrix} M_r(\ell v,lw_{2i+1},t),M_r(\ell v,Av,t)\\ M_r(\ell w_{2i+1},Bw_{2i+1},t),\\ M_r(\ell w_{2i+1},Av,t),M_r(\ell v,Bw_{2i+1},t)\end{Bmatrix}$$
$$=min\begin{Bmatrix} M_r(\ell v,lw_{2i+1},t),M_r(lv,Av,t)\\ M_r(\ell w_{2i+1},lw_{2i+2},t),M_r(\ell w_{2i+1},Av,t),M_r(\ell v,lw_{2i+2},t)\end{Bmatrix}$$
$$\to min\{0,M_r(lv,Av,t)\}=0 \; as\; i\to\infty.$$

$$=max\begin{Bmatrix} N_r(\ell v,lw_{2i+1},t),N_r(lv,Av,t)\\ N_r(\ell w_{2i+1},lw_{2i+2},t),N_r(\ell w_{2i+1},Av,t),N_r(\ell v,lw_{2i+2},t)\end{Bmatrix}$$
$$\to max\{0,N_r(lv,Av,t)\}=0 \; as\; i\to\infty.$$
$$U(A,B,l,v,w_{2i+1},t)=max\begin{Bmatrix} R_r(\ell v,lw_{2i+1},t),R_r(\ell v,Av,t)\\ R_r(\ell w_{2i+1},Bw_{2i+1},t),\\ R_r(\ell w_{2i+1},Av,t),R_r(\ell v,Bw_{2i+1},t)\end{Bmatrix}$$
$$=max\begin{Bmatrix} R_r(\ell v,lw_{2i+1},t),R_r(lv,Av,t)\\ R_r(\ell w_{2i+1},lw_{2i+2},t),R_r(\ell w_{2i+1},Av,t),R_r(\ell v,lw_{2i+2},t)\end{Bmatrix}$$
$$\to max\{0,R_r(lv,Av,t)\}=0 \; as\; i\to\infty. \qquad 51$$

Now from (50) and (51), for $t>0$, we have,
$\lim_{i\to\infty} inf\left(M_r(lw_{2i+2},Av,t)\right) for\ t>0$, $\lim_{i\to\infty} inf\left(N_r(lw_{2i+2},Av,t)\right) for\ t>0$
$\lim_{i\to\infty} inf\left(R_r(lw_{2i+2},Av,t)\right) for\ t>0.$ 52

By using the value (48) and (52) in (49) with limit $i\longrightarrow\infty$, we get that $\frac{1}{M_r(\ell v,Av,t)}=0, u=\ell v=Av\ for\ t>0,$
$N_r(\ell v,Av,t)=0\Rightarrow u=\ell v=Av\ for\ t>0$ and
$R_r(\ell v,Av,t)=0\Rightarrow u=\ell v=Av\ for\ t>0.$
Next, to prove that $u=\ell v=Bv$. Since, M_r,N_r,R_r is triangular,
$\frac{1}{M_r(\ell v,Bv,t)}-1\le\frac{1}{M_r(lv,lw_{2i+1},t)}-1+\frac{1}{M_r(lw_{2i+1},Bv,t)}-1, for\ t>0,$
$N_r(\ell v,Bv,t)\le N_r(lv,lw_{2i+1},t)+N_r(lw_{2i+1},Bv,t), for\ t>0$ and
$R_r(\ell v,Bv,t)\le R_r(lv,lw_{2i+1},t)+R_r(lw_{2i+1},Bv,t), for\ t>0.$ 53
Now, from (34), (46), (48), and by Definition 2.2 (iv), for $t>0$, we have that

$$\frac{1}{M_r(\ell w_{2i+1},Bv,t)}-1=\frac{1}{M_r(Aw_{2i},Bv,t)}-1$$
$$\le\begin{Bmatrix} a\left(M_r(\ell w_{2i},lv,t)\right)+b\left(U(A,B,l,w_{2i},v,t)\right)\\ +c\left(\frac{M_r(\ell w_{2i},Bv,2t).M_r(\ell w_{2i},\ell w_{2i},t).M_r(\ell v,Aw_{2i},2t)}{M_r(\ell w_{2i},Aw_{2i},t).M_r(\ell v,Bv,t)}\right)\end{Bmatrix}$$
$$\le\begin{Bmatrix} a\left(M_r(\ell w_{2i},\ell v,t)\right)+b\left(U(A,B,l,w_{2i},v,t)\right)\\ +c\left(\frac{M_r(\ell w_{2i},lv,t).M_r(\ell v,Bv,t).M_r(\ell w_{2i},\ell v,t).M_r(\ell v,\ell w_{2i+1},t)}{M_r(\ell w_{2i,}\ell w_{2i}+1,t).M_r(lv,Bv,t)}\right)\end{Bmatrix}$$
$$\to b\left(\frac{1}{U(A,B,l,v,w_{2i},t)}-1\right)\ as\ t\to\infty.$$
$$N_r(\ell w_{2i+1},Bv,t)=N_r(Aw_{2i},Bv,t)$$
$$\le\begin{Bmatrix} a\left(N_r(\ell w_{2i},\ell v,t)\right)+b\left(U(A,B,l,w_{2i},v,t)\right)\\ +c\left(\frac{N_r(\ell w_{2i},lv,t).N_r(\ell v,Bv,t).N_r(\ell w_{2i},\ell v,t).N_r(\ell v,\ell w_{2i+1},t)}{N_r(\ell w_{2i,}\ell w_{2i}+1,t).N_r(lv,Bv,t)}\right)\end{Bmatrix}$$
$$\to b(U(A,B,l,v,w_{2i},t))\ as\ t\to\infty.$$
$$R_r(\ell w_{2i+1},Bv,t)=R_r(Aw_{2i},Bv,t)$$
$$\le\begin{Bmatrix} a\left(R_r(\ell w_{2i},lv,t)\right)+b\left(U(A,B,l,w_{2i},v,t)\right)\\ +c\left(\frac{R_r(\ell w_{2i},Bv,2t).R_r(\ell w_{2i},\ell w_{2i},t).R_r(\ell v,Aw_{2i},2t)}{R_r(\ell w_{2i},Aw_{2i},t).R_r(\ell v,Bv,t)}\right)\end{Bmatrix}$$
$$\le\begin{Bmatrix} a\left(R_r(\ell w_{2i},\ell v,t)\right)+b\left(U(A,B,l,w_{2i},v,t)\right)\\ +c\left(\frac{R_r(\ell w_{2i},lv,t).R_r(\ell v,Bv,t).R_r(\ell w_{2i},\ell v,t).R_r(\ell v,\ell w_{2i+1},t)}{R_r(\ell w_{2i,}\ell w_{2i}+1,t).R_r(lv,Bv,t)}\right)\end{Bmatrix}$$
$$\to b(U(A,B,l,v,w_{2i},t))\ as\ t\to\infty. \qquad 54$$
Where,
$$U(A,B,l,w_{2i},v,t)=min\begin{Bmatrix} M_r(lw_{2i},lv,t),M_r(lw_{2i},Aw_{2i},t)\\ M_r(\ell v,Bv,t),M_r(lv,lA,t),M_r(lw_{2i},Bv,t)\end{Bmatrix}$$
$$=min\begin{Bmatrix} M_r(lw_{2i},lv,t),M_r(lw_{2i},lw_{2i+1},t)\\ M_r(lv,Bv,t),M_r(\ell v,lw_{2i+1},t),M_r(lw_{2i},Bv,t)\end{Bmatrix}$$

$$\rightarrow min\{0, M_r(lv, Bv, t)\} = 0 \; as \; i \rightarrow \infty$$

$$U(A, B, l, w_{2i}, v, t) = max\begin{Bmatrix} N_r(lw_{2i}, lv, t), N_r(lw_{2i}, Aw_{2i}, t) \\ N_r(\ell v, Bv, t), N_r(lv, lA, t), N_r(lw_{2i}, Bv, t) \end{Bmatrix}$$

$$= max\begin{Bmatrix} N_r(lw_{2i}, lv, t), N_r(lw_{2i}, lw_{2i+1}, t) \\ N_r(lv, Bv, t), N_r(\ell v, lw_{2i+1}, t), N_r(lw_{2i}, Bv, t) \end{Bmatrix}$$

$$\rightarrow max\{0, N_r(lv, Bv, t)\} = 0 \; as \; i \rightarrow \infty$$

$$U(A, B, l, w_{2i}, v, t) = max\begin{Bmatrix} R_r(lw_{2i}, lv, t), R_r(lw_{2i}, Aw_{2i}, t) \\ R_r(\ell v, Bv, t), R_r(lv, lA, t), R_r(lw_{2i}, Bv, t) \end{Bmatrix}$$

$$= max\begin{Bmatrix} R_r(lw_{2i}, lv, t), R_r(lw_{2i}, lw_{2i+1}, t) \\ R_r(lv, Bv, t), R_r(\ell v, lw_{2i+1}, t), R_r(lw_{2i}, Bv, t) \end{Bmatrix}$$

$$\rightarrow max\{0, R_r(lv, Bv, t)\} = 0 \; as \; i \rightarrow \infty. \qquad 55$$

Now from (54) and (55), for $t > 0$, $\lim_{i\to\infty} sup\big(M_r(lw_{2i+1}, Bv, t)\big) \; for \; t > 0$,

$\lim_{i\to\infty} inf\big(N_r(lw_{2i+1}, Bv, t)\big) \; for \; t > 0$ and

$\lim_{i\to\infty} inf\big(R_r(lw_{2i+1}, Bv, t)\big) \; for \; t > 0$. 56

By using the value (48) and (56) in (53) with $limit \; i \longrightarrow \infty$, we obtain

$M_r(\ell v, Bv, t) = 1 \Rightarrow u = \ell v = Bv \; for \; t > 0$.

$M_r(\ell v, Bv, t) = 0 \Rightarrow u = \ell v = Bv \; for \; t > 0$ and

$M_r(\ell v, Bv, t) = 0 \Rightarrow u = \ell v = Bv \; for \; t > 0$.

Hence, we obtain that $u$is a common coincidence point of the mappings ℓ, A, and B in Wsuch that $u = \ell v = Av = Bv$.

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